Goals
- Diagonalizability
- Eigenspaces
- Diagonalization algorithm

Recall $f : V \to V$ linear trans'n has eigenvector $v \neq 0$ with eigenvalue λ when $f(v) = \lambda v$.

Defn For $\dim V = n$, a linear trans'n $f : V \to V$ is **diagonalizable** when either of the following equivalent conditions holds:

- $\exists$ ordered basis $\alpha = (v_1, \dots, v_n)$ of V such that $A^\alpha_\alpha(f) = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ for some $\lambda_i \in F$

- f has a basis of eigenvectors.

E.g. • $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathbb{R}^{2\times 2}$ is diagonalizable iff

$\theta = n\pi$ for some $n \in \mathbb{Z}$.

• $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ is not diagonalizable for $\lambda \neq 0$.

Recall If $\alpha = (v_1, \dots, v_n)$ is a basis of eigenvectors of $A \in F^{n\times n}$ and $P = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$, then $D = P^{-1}AP$ is diagonal with eigenvalues on diagonal. Thus A is diagonal iff it is similar (conjugate) to a diagonal matrix.

chi

Recall $\chi_A(x) = \det(A - xI_n)$ is the characteristic polynomial of A and λ is an eigenvalue of A iff $\chi_A(\lambda) = 0$.

E.g. If $A = \begin{pmatrix} 2 & -7 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & 2 \end{pmatrix}$ then $\chi_A(x) = \det\begin{pmatrix} 2-x & -7 & 3 \\ 0 & -5-x & 3 \\ 0 & 0 & 2-x \end{pmatrix}$

$$= -(x-2)^2(x+5).$$

Thus A has eigenvalues $2, -5$.

with algebraic multiplicity 2

Defn Let λ be an eigenvalue of a matrix A. The eigenspace of A for λ is

$$E_\lambda(A) := \{v \in V \mid \underset{(A-\lambda I_n)v = 0}{Av = \lambda v}\} = \ker(A - \lambda I_n).$$

E.g. (ctd) To compute $E_2 = E_2(A)$:

$$A - 2I_3 = \begin{pmatrix} 2 & -7 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -7 & 3 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[\text{G-J}]{} \begin{pmatrix} 0 & 1 & -3/7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow E_2 = \ker(A - 2I_3) = \left\{ \left(x, \tfrac{3}{7}z, z\right) \,\middle|\, x, z \in \mathbb{R} \right\}$$

with basis $\left((1,0,0), \left(0, \tfrac{3}{7}, 1\right)\right)$

or $\left((1,0,0), (0,3,7)\right)$.

To compute $E_{-5} = E_{-5}(A)$:

$$A + 5I_3 = \begin{pmatrix} 7 & -7 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 7 \end{pmatrix} \underset{G\text{-}J}{\rightsquigarrow} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow E_{-5} = \ker(A + 5I_3) = \{(y, y, 0) \mid y \in \mathbb{R}\}$$

with basis $(1,1,0)$.

Fact Eigenvectors in distinct eigenspaces are lin ind. (Proved soon.)

So $((1,0,0), (0,3,7), (1,1,0))$ is a basis of eigenvectors for A.

Set $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 7 & 0 \end{pmatrix}$. Then $P^{-1}AP = \operatorname{diag}(2, 2, -5)$.

$$A = \begin{pmatrix} 2 & -7 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

<u>E.g.</u> Let's modify A slightly: $A' := \begin{pmatrix} 2 & 1 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & 2 \end{pmatrix}$.

This has the same char poly and same eigenvalues as A!

A basis for $E_{-5}(A')$ is $(-7, 1, 0)$. $\longrightarrow$ sim techniques as above

<u>Problem</u> Find a basis for $E_2(A')$.

$$A' - 2I_3 = \begin{pmatrix} 0 & 1 & 3 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[\text{G-J}]{} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \ker(A' - 2I_3) = \{(x, 0, 0) \mid x \in \mathbb{R}\}$$

with basis $(1,0,0)$

There are at most two lin ind eigenvectors for A'!

We conclude that A' does not admit a basis of eigenvectors

$\Rightarrow A'$ is not diagonalizable.

Diagonalization Algorithm

(1) Find eigenvalues as roots of χ_A.

(2) For each eigenvalue λ, compute a basis of $E_\lambda(A)$.

(3) The matrix A is diagonalizable iff the total number of basis vectors in (2) is n (for $A \in F^{n \times n}$).

If so, the vectors found are an eigenbasis of A, and if P is the matrix with columns these vectors, then

$$D = P^{-1} A P$$

is diagonal with eigenvalues on diagonal.

Defn If λ is an eigenvalue of A, its *algebraic multiplicity* is the # of $(x-\lambda)$'s in $\chi_A(x)$. The *geometric multiplicity* of λ is $\dim E_\lambda(A)$.

<u>Note</u> $\sum$ geom mults $\le \sum$ alg mults $\le n$

and A is diagonalizable iff <u>both</u> sums equal n

iff $\sum$ geom mults $= n$.

<u>IOU</u> Vectors from different eigenspaces are lin ind.

Determinants of endomorphisms

Know of det as
$$\begin{array}{ccc} F^{n\times n} & \longrightarrow & F \\ \cong & & \\ \mathrm{End}(F^n) & & \end{array}$$

$\underline{Q}$ Does $\det(f)$ make sense for $f: V \longrightarrow V$ linear?

A Yes! for V finite dim'l.

Define $\det(f)$ by choosing ordered basis of V, say α

Then f has a matrix $A_\alpha^\alpha(f) \in F^{n \times n}$.

Hope $\det(f) := \det A_\alpha^\alpha(f)$ — but need to show this does not depend on choice of basis!

Suppose β is some other ordered basis of V.

From hw, know there exists matrix P s.t.

$$A_\beta^\beta(f) = P^{-1} A_\alpha^\alpha(f) P.$$

Thus $\det A^{\beta}_{\beta}(f) = \det\left(P^{-1} A^{\alpha}_{\alpha}(f)\, P\right)$

$$= \det(P^{-1}) \det\left(A^{\alpha}_{\alpha}(f)\right) \det(P) \quad \text{[mult of det]}$$

$$= \det A^{\alpha}_{\alpha}(f) \cdot \cancel{\frac{\det(P)}{\det(P)}}^{\,1}$$

$$= \det A^{\alpha}_{\alpha}(f)$$

So the value of $\det(f)$ does not depend on choice of basis.

if $CD = I$, then

$\det(CD) = \det I$

$\det C \cdot \det D = 1$

$\Rightarrow \det D = \dfrac{1}{\det C}$

This allows us to extend the difn of characteristic polynomial: if $f: V \to V$ linear, $\dim V < \infty$

then $\chi_f(x) = \det(f - x \cdot id_V)$

i.e. choose basis α of V

$$\chi_f(x) = \det\left(A_\alpha^\alpha(f) - x \cdot I \right)$$

Then eigenvalues of $f \overset{\cong}{\longleftrightarrow}$ roots of $\chi_f(x)$.