

- Goals
- Eigenvectors with distinct eigenvalues are linearly independent
 - Algebraic & geometric multiplicity
 - Jordan normal form

Recall If $f: V \rightarrow V$ linear, then $\chi_f(x) = \det(A_\alpha^\alpha(f) - x \cdot I_n)$
for a some/any ordered basis of V and $n = \dim V$.

Prop Suppose $f: V \rightarrow V$ linear with eigenvectors $v_1, \dots, v_k \in V$
corresponding to eigenvalues $\lambda_1, \dots, \lambda_k$, $\lambda_i \neq \lambda_j$ for $i \neq j$.
Then $v_1, \dots, v_k$ are linearly independent.

Pf Proceed by induction on k .

$k=1$ Eigenvectors are nonzero. ✓

$k>1$ For induction, suppose $v_1, \dots, v_{k-1}$ are lin ind and that

$$\mu_1 v_1 + \dots + \mu_k v_k = 0. \quad \text{WTS } \mu_1 = \dots = \mu_k = 0$$

Apply $f - \lambda_k \text{id}_V$ to this expression:

$$(f - \lambda_k \text{id}_V)(\mu_1 v_1 + \dots + \mu_k v_k) = (f - \lambda_k \text{id}_V)(0)$$

$$\Rightarrow \mu_1 f(v_1) + \dots + \mu_k f(v_k) - (\mu_1 \lambda_k v_1 + \dots + \mu_k \lambda_k v_k) = 0$$

$$\Rightarrow \mu_1 \lambda_1 v_1 + \dots + \mu_k \lambda_k v_k - (\mu_1 \lambda_k v_1 + \dots + \mu_k \lambda_k v_k) = 0 \quad [v_i \text{ eigen for } f]$$

$$\Rightarrow (\mu_1 (\lambda_1 - \lambda_k) v_1 + \dots + \mu_{k-1} (\lambda_{k-1} - \lambda_k) v_{k-1}) + \cancel{\mu_k (\lambda_k - \lambda_k) v_k} = 0$$

$$\Rightarrow \mu_1 (\lambda_1 - \lambda_k) = \dots = \mu_{k-1} (\lambda_{k-1} - \lambda_k) = 0 \quad \rightarrow = 0 \quad [\text{lin ind } v_1, \dots, v_{k-1}]$$

$$\Rightarrow \mu_1 = \dots = \mu_{k-1} = 0 \quad [\lambda_i \text{ distinct}]$$

$$\Rightarrow \mu_k v_k = 0 \Rightarrow \mu_k = 0 \text{ too.}$$

Thus ^(by ~~sk~~) $v_1, \dots, v_k$ are lin ind, completing the induction. $\square$

Cor If $f: V \rightarrow V$ has $\dim V$ distinct eigenvalues, then f is diagonalizable.

2 With repeated eigenvalues, f might be diagonalizable, might not.
Note Now certain that diagonalization algorithm works!

Multiplicity

Defn A polynomial $p(x) \in F[x]$ splits over F when
 $\exists c, \lambda_1, \dots, \lambda_n \in F$ such that $p(x) = c(x - \lambda_1) \dots (x - \lambda_n)$.

E.g. $x^2 + 1$ splits over $\mathbb{C}$ but not over $\mathbb{R}$.
 $= (x+i)(x-i)$

Fundamental Theorem of Algebra Every $p(x) \in \mathbb{C}[x]$ splits over $\mathbb{C}$.

Prop Suppose $f: V \rightarrow V$ linear, $\dim V = n < \infty$. If f is diagonalizable, then $\chi_f(x)$ splits over F .

Pf If f is diagonalizable, then $\exists$ ordered basis α of V such that $A_\alpha(f) = D = \text{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_i \in F$.

$$\begin{aligned}\text{Thus } \chi_f(x) &= \det(D - xI_n) = (\lambda_1 - x) \cdots (\lambda_n - x) \\ &= (-1)^n (x - \lambda_1) \cdots (x - \lambda_n). \quad \square\end{aligned}$$

 2 The converse fails: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has characteristic polynomial $(x-1)^2$ but $\dim E_1 = 1$.

Defn Let $\dim V < \infty$. The geometric multiplicity of an eigenvalue λ of a linear transformation $f: V \rightarrow V$ is

$$\dim E_\lambda(f).$$

The algebraic multiplicity of λ is the number of times $(x-\lambda)$ divides $\chi_f(x)$.

Prop The geometric multiplicity of an eigenvalue of f is $\leq$ its algebraic multiplicity.

Pf Let $v_1, \dots, v_k$ be a basis of $E_\lambda(f)$, extend to a basis $v_1, \dots, v_n$ of V .

The matrix for f wrt this basis looks like

$$A = \left(\begin{array}{c|c} \lambda I_k & B \\ \hline 0 & C \end{array} \right).$$

$$f(v_i) = \lambda v_i \text{ for } 1 \leq i \leq k$$

Thus $\chi_f(x) = \det \left(\begin{array}{c|c} (\lambda-x)I_k & B \\ \hline 0 & C-xI_{n-k} \end{array} \right)$ induction + Laplace expansion:

$= (\lambda-x)^k \chi_c(x)$ ←



$$\left(\begin{array}{c|c} \lambda-x & \\ \hline & \lambda-x & B \\ & \ddots & \\ & \lambda-x & \\ \hline & & C-xI \end{array} \right)$$

so $k = \dim E_\lambda(f) \leq$ algebraic multiplicity of λ .

Jordan normal form

$1 \leq \text{geom mult} \leq \text{alg mult}$

↑ need = for diagonalizability

A Jordan block of size k for $\lambda \in F$ is the $k \times k$ matrix

$$J_k(\lambda) := \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

E.g. $J_4(3) = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad J_1(\lambda) = (\lambda)$

A matrix is in Jordan form when it is a "block sum" of Jordan blocks:

$$J_{k_1}(\lambda_1) \oplus \dots \oplus J_{k_r}(\lambda_r) = \begin{pmatrix} J_{k_1}(\lambda_1) & & \\ & J_{k_2}(\lambda_2) & \\ & & \ddots \\ & & & J_{k_r}(\lambda_r) \end{pmatrix}$$

E.g.

$$J_2(2) \oplus J_2(2) \oplus J_1(5) \oplus J_3(4) = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & \\ & & 5 & \\ & & & \boxed{\begin{matrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{matrix}} \end{pmatrix}$$

Thm Suppose $\dim V < \infty$, $f: V \rightarrow V$ linear with $\chi_f(x)$ split over F .

Then $\exists$ ordered basis α of V such that $A_\alpha^\sim(f)$ is in Jordan form, the Jordan form of f is unique up to permutation of Jordan blocks.

PF Use the structure theorem for finitely generated modules over a principal ideal domain — Math 332. $\square$

Note f is diagonalizable iff all its Jordan blocks have size 1.

$$U_n(\cos x) = \cos(nx)$$

Chebyshev polynomial
of 1st type

$$U_n(x) = \det \begin{pmatrix} x & 1 & & & \\ & 2x & 1 & & \\ & & 2x & 1 & \\ & & & \ddots & 2x \end{pmatrix}$$