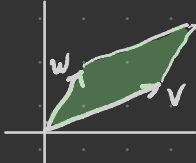
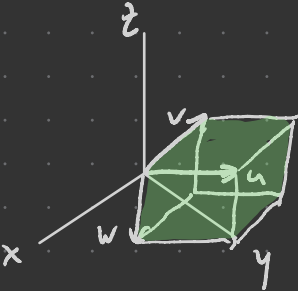


- Goals
- signed volume via det
 - signed volume and linear transformations.

Given vectors $v_1, \dots, v_n \in \mathbb{R}^n$, the parallelepiped spanned by $v_1, \dots, v_n$ is

$$P(v_1, \dots, v_n) = \{ t_1 v_1 + \dots + t_n v_n \mid 0 \leq t_i \leq 1 \}.$$

E.g. • In $\mathbb{R}^2$, $P(v, w) =$ 

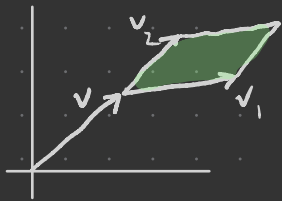
• In $\mathbb{R}^3$, $P(u, v, w) =$ 

• In $\mathbb{R}^n$, $B_n = P(e_1, e_2, \dots, e_n) = [0, 1]^n$ is the unit box.

Defn A parallelepiped in $\mathbb{R}^n$ is any set of the form

$$P(v_1, \dots, v_n) + v = \{t_1 v_1 + \dots + t_n v_n + v \mid 0 \leq t_i \leq 1\}.$$

E.g.



Note If $A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_n \\ | & | & | \end{pmatrix} \in \mathbb{R}^{n \times n}$, then $A \cdot B_n = P(v_1, \dots, v_n)$.

Indeed, $A \cdot (t_1 e_1 + t_2 e_2 + \dots + t_n e_n) = t_1 v_1 + \dots + t_n v_n$.

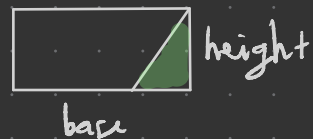
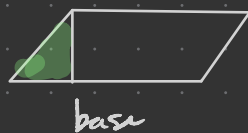
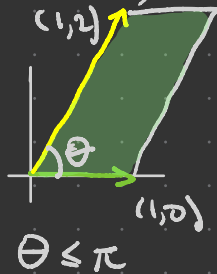
Signed area

$$SA: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(v, w) \longmapsto \begin{cases} \text{area}(P(v, w)) & \text{if ccw angle } v \text{ to } w \text{ is } \leq \pi \\ -\text{area}(P(v, w)) & \text{if ccw angle } v \text{ to } w > \pi \end{cases}$$



E.g. $SA((1,0), (1,2)) = \text{base} \times \text{height} = 2$



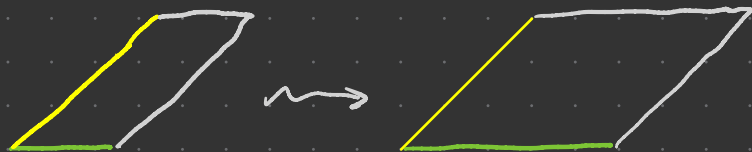
Claim SA is multilinear, alternating, normalized

and thus $SA(v, w) = \det \begin{pmatrix} -v- \\ -w- \end{pmatrix} = \det \begin{pmatrix} v & w \\ 1 & 1 \end{pmatrix}$

Pf • $SA(e_1, e_2) = \text{area} \left(\begin{array}{|c|} \hline \square \\ \hline \end{array} \right) = 1$ so SA is normalized

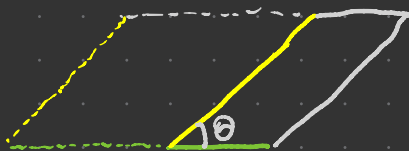
• $SA(v, v) = \text{area} \left(\begin{array}{|c|} \hline \triangle \\ \hline \end{array} \right) = 0$ so SA is alternating.

• For $\lambda > 0$, $SA(\lambda v, w) = \lambda SA(v, w)$:



same angle
base scaled by $\lambda > 0$

For $\lambda < 0$



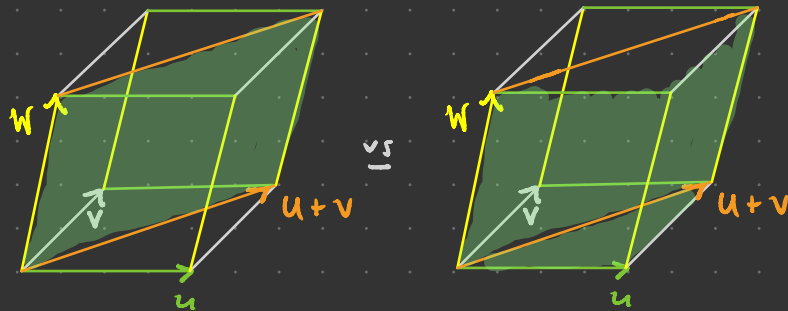
angle from θ to $\pi + \theta$

base scaled by $|\lambda|$

so $SA(\lambda v, w) = \lambda SA(v, w)$.

Similar arguments give $SA(v, \lambda w) = \lambda SA(v, w)$.

• $SA(u+v, w) = SA(u, w) + SA(v, w)$



implies $SA(u+v, w) = SA(u, w) + SA(v, w)$

(and similarly in 2nd variable).



Defn The volume of $P(v_1, \dots, v_n)$ is

$$\left| \det \begin{pmatrix} | & | & | \\ v_1 & \dots & v_n \\ | & | & | \end{pmatrix} \right|.$$

Thm For any parallelepiped $Q = P(v_1, \dots, v_n)$ and linear transformation $L: \mathbb{R}^n \rightarrow \mathbb{R}^n$ induced by a matrix A , $L(Q) = P(Av_1, \dots, Av_n)$ and $\text{vol}(LQ) = |\det A| \text{vol}(Q)$.

Pf Let $B = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}$. Then $Q = B \cdot B_n$ and

$$LQ = A \cdot (B \cdot B_n) = (AB) \cdot B_n \text{ with } AB = \begin{pmatrix} | & & | \\ Av_1 & \dots & Av_n \\ | & & | \end{pmatrix}$$

so $LQ = P(Av_1, \dots, Av_n)$. Now

$$\text{vol}(LQ) = |\det(AB)| \overset{\text{det multiplicative}}{=} |\det(A) \cdot \det(B)|$$

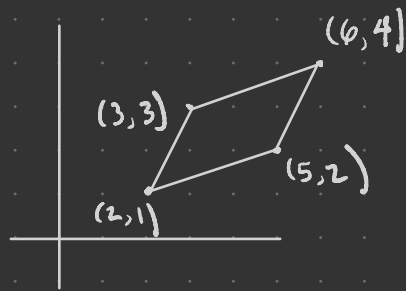
$$= |\det A| |\det B| = |\det A| \text{vol}(Q) \quad \square$$

It is multiplicative $\underbrace{\quad}_{\text{vol}(Q)}$

Rmk We proved that L transforms volumes of parallelepipeds by $|\det A|$ (for $L = \text{map}_A$).

In Math 202, you will generalize this to other volumes.

Problem Find the area of this figure:



Answer This is $P((3,1), (1,2)) + (2,1)$ so its area is

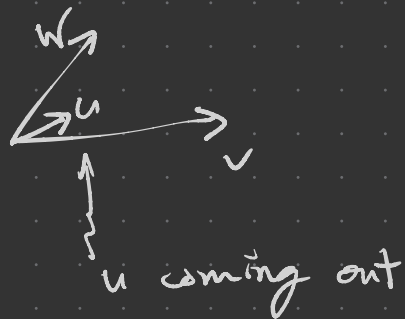
$$\left| \det \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \right| = |6-1| = 5.$$

Problem What is the volume of λB_n , $\lambda > 0$?

Answer $\text{vol}(\lambda B_n) = \left| \det \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} \right| = |\lambda^n| = \lambda^n$

Orientation Call an ordered basis $(v_1, \dots, v_n)$ of $\mathbb{R}^n$ positively oriented when $\det \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix} > 0$;
if $\det \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix} < 0$ call it negatively oriented.

Note $\Sigma_n = (e_1, \dots, e_n)$ is positively or'd



(v, w, u) is pos or'd in $\mathbb{R}^3$

In 2D, angle condition.