

- Goals
- Laplace expansion of det
 - Existence & uniqueness of det

Defn For $A \in F^{n \times n}$, $1 \leq i, j \leq n$, set $A^{ij} \in F^{(n-1) \times (n-1)}$ to be the matrix created by deleting the i -th row and j -th column from A .

Eg. If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$, then $A^{23} = \begin{pmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{pmatrix}$ not 23rd power

Thm (Laplace expansion) Suppose $A \in F^{n \times n}$ and $1 \leq k \leq n$. Then

$$\det A = \sum_{j=1}^n (-1)^{k+j} A_{kj} \det(A^{kj})$$

(k,j) minor
 with sign (k,j) cofactor

expansion of $\det A$ along the k -th row of A

E.g.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (-1)^{1+1} a \det(d) + (-1)^{1+2} b \det(c) \\
 = ad - bc \quad \checkmark$$

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 5 & 6 & 7 \end{pmatrix} = (-1)^{2+1} \cdot 0 \det \begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix} + (-1)^{2+2} \cdot 4 \det \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} \\
 + (-1)^{2+3} \cdot 0 \cdot \det \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$$

$$= 4(1.7 - 3.5) = -32.$$

Note $(-1)^{k+j}$ makes a "sign checkerboard":

$$\begin{pmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This can be a helpful mnemonic when expanding along some row.

Cor [Laplace expansion along columns] For $A \in F^{n \times n}$, $1 \leq k \leq n$,

$$\det A = \sum_{i=1}^n (-1)^{i+k} A_{ik} \det(A^{ik})$$

Pf We know $\det A = \det A^T$ and this is Laplace exp'n along the k -th row of A^T .

Question What is the determinant of $\begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & 6 \\ 4 & 7 & 0 \end{pmatrix}$?

$$-6 \cdot \det \begin{pmatrix} 1 & 3 \\ 4 & 7 \end{pmatrix} = -6 \cdot (-5) = 30$$

Pf sketch for Laplace Expansion Thm By permutation exp'n,

$$\det A = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n A_{i, \sigma(i)}.$$

Factor out all the A_{kj} terms:

$$\det A = \sum_{j=1}^n A_{kj} \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma(k)=j}} \operatorname{sgn}(\sigma) \prod_{\substack{1 \leq i \leq n \\ i \neq k}} A_{i, \sigma(i)}$$

Need to show this is
 $(-1)^{k+j} \det A_{ij}$

The  part has all the correct A_{ij} terms. Must check that the sign is off by $(-1)^{k+j}$ (hint: swap rows). $\square$

Thm $[\exists! \det]$ There is a unique multilinear, alternating, normalized function $\det: F^{n \times n} \rightarrow F$.

Pf Inspired by Laplace expansion, make the following

recursive definition of a function $d: F^{n \times n} \rightarrow F$:

$$\underline{n=1} \quad d(a) = a$$

$$\underline{n>1} \quad d(A) = \sum_{j=1}^n (-1)^{1+j} A_{1j} d(A^{(1j)})$$

Check d is multilin, alt, normalized.

Then d is a well-defined determinant function.

Our work with $\det$ & row operations shows that all determinant functions are determined by a sequence of row operations to rref:

if $E_1, \dots, E_k$ elementary s.t. $\text{rref}(A) = E_k \cdots E_1 A$

$$\text{then } \det A = \frac{\det(\text{rref}(A))}{\det(E_k) \cdots \det(E_1)}$$

Thus any two det functions are in fact the same! $\square$

Time permitting, the general linear group:

Let $F^* := F \setminus \{0\}$, $GL_n(F) \subseteq F^{n \times n}$ be the invertible $n \times n$ matrices.

Then $GL_n(F) = \det^{-1} F^* = \{A \in F^{n \times n} \mid \det A \in F^*\}$

$$\begin{array}{ccc} F^{n \times n} & \xrightarrow{\det} & F \\ \cup & & \cup \\ GL_n(F) & \xrightarrow{\det} & F^* \end{array} \quad \begin{array}{c} \Downarrow \\ \det A \neq 0 \end{array}$$

(group under matrix multiplication

#3

$$\mathbb{R}^2 \xrightarrow{t} \mathbb{R}^2$$

$$e_1 \longmapsto (1,1) + 3(1,-2) \\ = (4,-5)$$

$$e_2 \longmapsto 2(1,1) + 4(1,-2) \\ = (6,-6)$$

$$A_{\varepsilon}^{\rho}(t) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\beta = ((1,1), (1,-2))$$

$$\Rightarrow A_{\varepsilon}^{\varepsilon}(t) = \begin{pmatrix} 4 & 6 \\ -5 & -6 \end{pmatrix}$$

$$\delta = ((0,1), (1,1)) \quad \gamma = ((-1,0), (2,1))$$

$$\varepsilon = ((1,2), (1,0)) \quad \zeta = ((1,2), (2,1))$$

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{t} & \mathbb{R}^2 \\ \text{Rep}_{\delta} \downarrow \uparrow P & & Q \uparrow \downarrow \text{Rep}_{\gamma} \\ \mathbb{R}^2 & \xrightarrow{A_{\delta}^{\gamma}(t)} & \mathbb{R}^2 \end{array}$$

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A_{\delta}^{\eta}(t) = Q^{-1} A_{\xi}^{\xi}(t) P$$

$$= \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ -5 & -6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 10 \\ -6 & -11 \end{pmatrix}$$

$$= \begin{pmatrix} -18 & -32 \\ -6 & -11 \end{pmatrix}$$

$$t(0,1) = (6, -6)$$

$$t(1,1) = t(e_1) + t(e_2)$$

$$= (4, -5) + (6, -6)$$

$$= (10, -11)$$

$$-18 \cdot (-1, 0) - 6(2, 1) = (6, -6) \quad \checkmark$$

$$-32 \cdot (-1, 0) - 11(2, 1) = (10, -11) \quad \checkmark$$