

- Goals
- Define eigenvectors & eigenvalues of linear trans'ns
 - Define the characteristic polynomial
 - $\{\text{eigenvalues}\} = \{\text{roots of characteristic polynomial}\}$

Defn Let $f: V \rightarrow V$ be a linear transformation. A nonzero vector $v \in V$ is an eigenvector of f with eigenvalue $\lambda \in F$ when

$$f(v) = \lambda v.$$

If $\alpha = (v_1, \dots, v_n)$ is a basis of eigenvectors of f with eigenvalues $\lambda_1, \dots, \lambda_n$, then

$$A_{\alpha}^{\alpha}(f) = \text{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

Diagonal matrices are much easier to work with, so this is a big win!

2 Not every lin trans'n has an associated basis of eigenvectors.

Problem Compute the powers of $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. What about $\text{diag}(\lambda_1, \dots, \lambda_n)^k$?

Answer

$$\text{diag}(\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k)$$

E.g. Let $A = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}$ with assoc lin trans'n $\text{map}_A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

$$\text{Since } \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} = 2 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\text{and } \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

we know $(2,3)$ and $(1,2)$ are eigenvectors of map_A with eigenvalues 2, 3, resp.

Let $\alpha = ((2,3), (1,2))$. Since $\text{map}_A(2,3) = 2 \cdot (2,3)$
 $\text{map}_A(1,2) = 3 \cdot (1,2)$,

we have $A_\alpha^\alpha(\text{map}_A) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} =: D$.

The matrix for Rep_α^{-1} is $P = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ \text{Rep}_\alpha \downarrow & & \downarrow \text{Rep}_\alpha \\ \mathbb{R}^2 & \xrightarrow{D} & \mathbb{R}^2 \end{array}$$

$$\Rightarrow D = P^{-1}AP.$$

Summary Eigenvectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are columns of P . Then

$D = P^{-1}AP$ is diagonal with eigenvalues on the diagonal.

In general, if $f: F^n \rightarrow F^n$ is a lin trans'n and $\alpha = (v_1, \dots, v_n)$ is an ordered basis of eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_n$,

$$P = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}, \quad A = \begin{pmatrix} | & & | \\ f(v_1) & \dots & f(v_n) \\ | & & | \end{pmatrix}, \quad \text{then}$$

$A_{\Sigma}(f)$

$$D = P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n).$$

How to find eigenvectors & eigenvalues:

For $A \in F^{n \times n}$, $v \in F^n$, $\lambda \in F$,

$$Av = \lambda v \iff (A - \lambda I_n)v = 0$$

$$\iff v \in \ker(A - \lambda I_n).$$

Thus λ is an eigenvalue of A iff $\ker(A - \lambda I_n) \neq 0$.

$$\text{But } \ker(A - \lambda I_n) \neq 0 \iff \text{rank}(A - \lambda I_n) < n$$

$$\iff \det(A - \lambda I_n) = 0.$$

E.g. $\det\left(\begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} -1-\lambda & 2 \\ -6 & 6-\lambda \end{pmatrix}$

$$= (-1-\lambda)(6-\lambda) + 12$$

$$= \lambda^2 - 5\lambda + 6$$

$$= (\lambda - 2)(\lambda - 3).$$

Thus the eigenvalues of $\begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}$ are $\lambda = 2, 3$.

How do we find eigenvectors?

Need to compute $\ker(A - \lambda I_n)$ for eigenvalues λ .

E.g. $A - 2I_2 = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$$= \begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix} \xrightarrow{\text{G-J}} \begin{pmatrix} 1 & -2/3 \\ 0 & 0 \end{pmatrix}$$

$$x \cdot 1 - \frac{2}{3}y = 0$$
$$x = \frac{2}{3}y$$

$$\begin{aligned} \text{So } \ker(A - 2I_2) &= \left\{ \left(\frac{2}{3}y, y \right) \mid y \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \left(\frac{2}{3}, 1 \right) \right\} = \text{span} \left\{ (2, 3) \right\} \end{aligned}$$

Similarly,

$$A - 3I_2 = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix} \xrightarrow{G-J} \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$$

$$\text{so } \ker(A - 3I_2) = \text{span}\{(1/2, 1)\} = \text{span}\{(1, 2)\}$$

Problem Find a matrix that is not diagonalizable. (Think geometrically.)

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ rotates by } \pi/2 \text{ ccw}$$

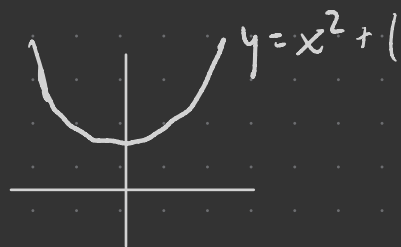
Defn $\chi_A(x) = \det(A - xI_n)$ is the characteristic polynomial of $A \in F^{n \times n}$.

Then The roots of χ_A are the eigenvalues of A .

$$\chi_{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}(x) = \det \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} -x & -1 \\ 1 & -x \end{pmatrix}$$

$$= x^2 - (-1) = x^2 + 1$$



has no roots over $\mathbb{R}$!

Does have roots over $\mathbb{C}$:

$$\lambda = \pm\sqrt{-1} = \pm i$$

$$\ker \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \right) = \ker \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \xrightarrow{r_1 \rightarrow ir_1} \begin{pmatrix} 1 & -i \\ 1 & -i \end{pmatrix} \xrightarrow{r_2 \rightarrow r_2 - r_1} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$$

$$\ker \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - iI_2 \right) = \text{span} \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$$

Check $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} = i \begin{pmatrix} i \\ 1 \end{pmatrix}$

Exc check other.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

"2x2 Jordan
block"

$$\chi_A(x) = \det \begin{pmatrix} 1-x & 1 \\ 0 & 1-x \end{pmatrix}$$

$$= (1-x)^2$$

$\Rightarrow \lambda = 1$ is the unique eigenvalue.

e_1 is an eigenvector but no others in

$F^2 = \text{span}\{e_1\}$! So not diagonalizable.