

- Goals
- $\det : F^{n \times n} \rightarrow F$ as an alternating, multilinear function such that $\det(I_n) = 1$.
 - Compute $\det$ via row ops
 - $\det A \neq 0 \iff \text{rank } A = n \iff A \text{ invertible}$

Hunting for a function $\det : F^{n \times n} \rightarrow F$ with some special properties. If $A \in F^{n \times n}$ has rows $r_1, \dots, r_n \in F^n$, write $\det A = \det(r_1, \dots, r_n)$.

Desiderata:

Multilinear: $\det$ is linear in each of its variables

Alternating: If $r_i = r_j$ for some $i \neq j$, then $\det(r_1, \dots, r_n) = 0$

Normalized: $\det(I_n) = \det(e_1, \dots, e_n) = 1$.

$$\det(\lambda r_i + r'_i, r_1, \dots, r_n) = \lambda \det(r_1, \dots, r_n) + \det(r'_i, \dots, r_n)$$

Thm For each $n \geq 0$, there is a unique determinant function satisfying these properties.

Proof — later! For now, derive consequences of properties.

Prop [det & row ops]

(1) If $A \xrightarrow{r_i \leftrightarrow r_j} B$, then $\det B = -\det A$

(2) If $A \xrightarrow{r_i \rightarrow \lambda r_i} B$, then $\det B = \lambda \det A$

(3) If $A \xrightarrow{r_i \rightarrow r_i + \lambda r_j} B$, then $\det B = \det A$

Alternative defn of alternating —
but doesn't work
for char $F = 2$
e.g. $\mathbb{Z}/2\mathbb{Z}$

Pf Sketch (1) $0 = \det(r_1 + r_2, r_1 + r_2)$

$$= \det(\cancel{r_1, r_1}) + \det(r_1, r_2) + \det(r_2, r_1) + \det(\cancel{r_2, r_2})$$

$$= 0 + \det A + \det B + 0 = \det A + \det B$$

(2) Multilinearity.

$$(3) \det(r_1, \lambda r_1 + r_2) = \lambda \det(r_1, r_1) + \det(r_1, r_2) \\ = 0 + \det(r_1, r_2)$$

□

E.g. $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det((a, b), (c, d))$

$$= a \det(e_1, (c, d)) + b \det(e_2, (c, d))$$
$$= a c \det(e_1, e_1) + a d \det(e_1, e_2) \\ + b c \det(e_2, e_1) + b d \det(e_2, e_2)$$
$$= a d \cdot \det I_2 - b c \cdot \det I_2$$
$$= ad - bc.$$

Note In general, we can use G-J reduction to compute determinants.

E.g.

E.g. $\det \begin{pmatrix} 4 & 2 & -1 & 8 \\ 0 & 5 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

$$= 4 \cdot 5 \cdot 2 \cdot 3 \det \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= 120 \det I_4$$

$$= 120$$

$$\det(A) = \det \begin{pmatrix} 1 & 2 & -2 \\ 9 & 4 & 0 \\ 2 & 2 & 4 \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 2 & -2 \\ 0 & -14 & 18 \\ 0 & -2 & 8 \end{pmatrix}$$

$$= -\det \begin{pmatrix} 1 & 2 & -2 \\ 0 & -2 & 8 \\ 0 & -14 & 18 \end{pmatrix}$$

$$= 2 \det \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & -14 & 18 \end{pmatrix}$$

$$= 2 \det \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & -38 \end{pmatrix}$$

$$= 2(-38) \det \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 2(-38) \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 2(-38) = -76.$$

Prop The determinant of an upper triangular matrix $\begin{pmatrix} * & * & * & \dots & * \\ 0 & * & * & \dots & * \\ 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & * \end{pmatrix}$

$= \begin{pmatrix} a_1 & a_2 & * \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_n \end{pmatrix}$ is $a_1 \dots a_n$, the product of its diagonal entries.

Pf The above argument generalizes as long as there are no 0's on the diagonal. In general, $\det A = \lambda \det(\text{rref}(A))$ for some $\lambda \in F \setminus \{0\}$ by G-J reduction. If A has a 0 on its diagonal, then $\text{rref}(A)$ has a row of all 0's and — pulling a 0 scalar out — we get $\det A = 0$. $\square$

Multilinearity U, V, W F -vs's

$f: V \times W \longrightarrow U$ is bilinear when

each function $f|_{V \times \{v\}}: V \longrightarrow U$

$f|_{\{v\} \times W}: W \longrightarrow U$

is linear.

Multilinear is this with more variables: $f: V_1 \times \dots \times V_n \longrightarrow U$.

For det, write $F^{n \times n} = \underbrace{F^n \times F^n \times \dots \times F^n}_{n \text{ times}}$

Prop For $A \in F^{n \times n}$, TFAE :

(1) $\det A \neq 0$, (2) $\text{rank } A = n$, (3) A is invertible.

Pf Already know (2) $\Leftrightarrow$ (3), so suffices to check (1) $\Leftrightarrow$ (2).

Since $\det A = \lambda \det(\text{rref}(A))$ for some $\lambda \in F - \{0\}$, we know $\det A = 0$

$\Leftrightarrow \det(\text{rref}(A)) = 0$. The rank of A is n iff $\text{rref}(A) = I_n$

iff $\det A = \lambda \neq 0$. $\square$

when $\text{rref}(A) \neq I_n$, it is upper triangular with a row of 0's
so $\det = 0$.

Preview

(1) $\det A^T = \det A$

(2) $\det AB = \det A \cdot \det B$

(3) "Laplace expansion" of $\det$ along any row or column

(4) "Permutation expansion" $\text{sign} = \pm 1$

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}$$

$n!$ terms $\xrightarrow{\sigma \in \mathfrak{S}_n} \mathfrak{S}_n$

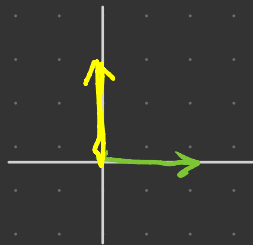
thus $\det A$
is a homogeneous
degree n polynomial
in entries of A

(5) Over $\mathbb{R}$, $\det A$ = "signed volume" of the parallelepiped
spanned by the columns of A .

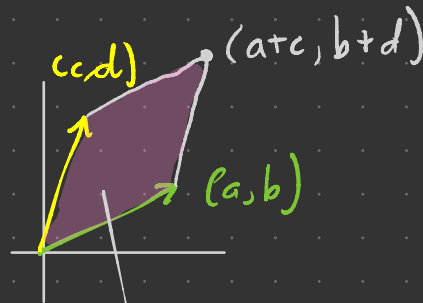
(6) $\det$ exists and is unique!

$$\mathfrak{S}_n = \{ \sigma : \underline{n} \rightarrow \underline{n} \mid \sigma \text{ bij} \}$$
$$\underline{n} = \{1, 2, \dots, n\}$$

$n=2$



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$



parallelogram spanned
by col's of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Get -1th volume
when A reverses
orientation

$$\text{Claim } \text{vol} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|$$
$$= |ad - bc|$$

$$\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 \cdot 0 - 1 \cdot 1 = -1$$