

Thus $\text{sgn}(\sigma) = (-1)^{\underbrace{\# \text{transpositions for } \sigma}_{\substack{\text{takes many values,} \\ \text{but all have same parity}}}} \in \{\pm 1\}$

E.g. $\begin{array}{ccc} 1 & & 1 \\ 2 & \times & 2 \\ 3 & \times & 3 \end{array} = \begin{array}{ccc} 1 & \rightarrow & 1 \\ 2 & \times & 2 \\ 3 & \times & 3 \end{array} \rightarrow \begin{array}{ccc} 1 & & 1 \\ 2 & \times & 2 \\ 3 & \times & 3 \end{array}$ is even.

24. X. 18

Thm (Permutation expansion) For $A \in F^{n \times n}$,

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$$

Thus $\det A$ is a homogeneous degree n polynomial in the entries of A .

E.g.

1 → 1 2 → 2 3 → 3	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$a_{11}a_{22}a_{33}$
1 ↘ 1 2 → 2 3 → 3	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$-a_{12}a_{21}a_{33}$
1 ↘ 1 2 ↘ 2 3 → 3	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$-a_{13}a_{22}a_{31}$
1 → 1 2 ↘ 2 3 → 3	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$-a_{11}a_{23}a_{32}$
1 ↘ 1 2 ↘ 2 3 ↘ 3	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$a_{12}a_{23}a_{31}$
1 ↘ 1 2 ↘ 2 3 ↘ 3	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$a_{13}a_{21}a_{32}$

so $\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \text{sum of}$ 

Exc Check that
permutation expansion
gives $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
 $= ad - bc$

Pf of Thm We want to compute

$$\det A = \det(A_{11}e_1 + A_{12}e_2 + \dots + A_{1n}e_n, \dots, A_{n1}e_1 + A_{n2}e_2 + \dots + A_{nn}e_n)$$

Expanding by multilinearity, we get n^n terms that look like

$$A_{1j_1} A_{2j_2} \dots A_{nj_n} \det(e_{j_1}, e_{j_2}, \dots, e_{j_n}).$$

But if $j_k = j_l$ for any $k \neq l$, then e_{j_k} will be duplicated in the det expression, which will thus be 0 by the alternating property.

As such, the only possible contributors to $\det A$ are of the form

$$A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)} \det(\underbrace{e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n)}}_{\text{rows of } P_\sigma})$$

$$= \text{sgn}(\sigma) \prod_{i=1}^n A_{i, \sigma(i)}$$

for $\sigma \in \mathcal{S}_n$. $\square$

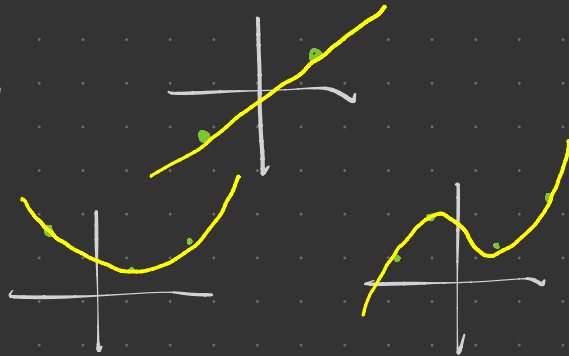
Polynomial interpolation

Suppose we have n points $(x_1, y_1), \dots, (x_n, y_n) \in \mathbb{R}^2$.

We expect there is a degree $n-1$ polynomial interpolating between the points:

$$p(x) = a_0 + a_1 x + \dots + a_{n-1} x^{n-1}$$

with $p(x_i) = y_i$ for $1 \leq i \leq n$.



To find $a_0, \dots, a_{n-1}$, consider the augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} & y_n \end{array} \right)$$

Vandermonde matrix $V \in \mathbb{R}^{n \times n}$

The system $p(x_i) = y_i$, $1 \leq i \leq n$, has a solution iff $\det V \neq 0$.

Can we compute $\det V$? Yes — via change-of-basis!

Consider the linear transformation

$$f: \mathbb{R}[x]_{\leq n-1} \rightarrow \mathbb{R}^n$$
$$p(x) \mapsto (p(x_1), \dots, p(x_n)).$$

$$\text{Let } \mu = (1, x, \dots, x^{n-1}), \quad \Sigma = (e_1, \dots, e_n).$$

$$\text{Then } A_{\mu}^{\Sigma}(f) = V.$$

Now consider a new ordered basis

$$\alpha = (1, x-x_1, (x-x_1)(x-x_2), \dots, (x-x_1)(x-x_2) \cdots (x-x_{n-1}))$$

of $\mathbb{R}[x]_{\leq n-1}$.

Check Since the i -th term of α is monic of degree i (i.e. $x^i + (\text{lower order terms})$), $A_\alpha^M(\text{id}_{\mathbb{R}[x]_{\leq n-1}})$ is upper triangular with 1's on the diagonal.

$$\text{Thus } \det A_\alpha^M(\text{id}_{\mathbb{R}[x]_{\leq n-1}}) = 1.$$

$$\text{We have } A_\alpha^\Sigma(f) = A_\mu^\Sigma(f) A_\alpha^M(\text{id}) = V \cdot A_\alpha^M(\text{id})$$

$$\Rightarrow \det A_\alpha^\Sigma(f) = \det V.$$

Evaluating the α polynomials at $x_1, \dots, x_n$, we also have

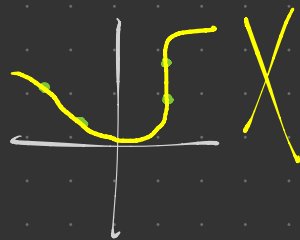
$$A_{\alpha}^z(f) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & x_2 - x_1 & 0 & \dots & 0 \\ 1 & x_3 - x_1 & (x_3 - x_2)(x_3 - x_1) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n - x_1 & (x_n - x_1)(x_n - x_2) & \dots & (x_n - x_1)(x_n - x_2) \dots (x_n - x_{n-1}) \end{pmatrix}$$

which is lower triangular with determinant the product of its diagonal entries. Thus

$$\det V = \prod_{1 \leq i < j \leq n} (x_j - x_i),$$

We have $\det V = 0$ iff some $x_i = x_j$, $i \neq j$.

If $\det V \neq 0$, then $V^{-1} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix}$ where



$p(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ interpolates between the points (x_i, y_i) .

Note $(\det V)^2 =:$ discriminant of a polynomial p with distinct roots $x_1, \dots, x_n$ — important in Galois theory.