Recall If V, W have finite ordered bases $\alpha = (v_1, \dots, v_n)$ and $\beta = (w_1, \dots, w_m)$, then

$$A_\alpha^\beta : \mathrm{Hom}(V,W) \xrightarrow{\cong} F^{m\times n}$$
$$f \longmapsto A_\alpha^\beta(f) = \begin{pmatrix} | & & | \\ \mathrm{Rep}_\beta v_1 & \cdots & \mathrm{Rep}_\beta v_n \\ | & & | \end{pmatrix}.$$

Furthermore, if $\underset{\alpha}{U} \xrightarrow{f} \underset{\beta}{V} \xrightarrow{g} \underset{\gamma}{W}$ ordered bases are linear transformations, then

then $$A_\alpha^\gamma(g\circ f) = A_\beta^\gamma(g) \cdot A_\alpha^\beta(f)$$

where the (i,j)-entry of $A\cdot B$ is the dot product of the i-th row of A with the j-th column of B.

$U \xrightarrow{h} V \underset{g'}{\overset{g}{\rightrightarrows}} W \underset{f'}{\overset{f}{\rightrightarrows}} T$ linear

Matrices

Observe: $(f \circ g) \circ h = f \circ (g \circ h) \implies (A \cdot B) \cdot C = A \cdot (B \cdot C)$

$(f + f') \circ g = (f \circ g) + (f' \circ g) \implies (A + A') \cdot B = A \cdot B + A' \cdot B$

$f \circ (g + g') = (f \circ g) + (f \circ g') \implies A \cdot (B + B') = A \cdot B + A \cdot B'$

Upshot $\underbrace{\mathrm{End}(V)}_{\text{endomorphisms}} := \mathrm{Hom}(V, V) \cong \underbrace{F^{n \times n}}_{\text{square matrices}}$

$F^{n \times n}$ form a ring — like a field but no inverses

If $V = F^n$, $W = F^m$ with standard ordered basis $\mathcal{E}_n = (e_1, \dots, e_n)$ and $\mathcal{E}_m = (e_1, \dots, e_m)$, get

$$\mathrm{Hom}(F^n, F^m) \cong F^{m \times n}$$

$$f \longmapsto A(f) := A^{\mathcal{E}_n}_{\mathcal{E}_m}(f) = \begin{pmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{pmatrix}$$

and we can be very explicit about the inverse:

$$\begin{tikzcd} \mathrm{Hom}(F^n, F^m) & F^{m \times n} \arrow[l] \\ x & F^n \arrow[d, "\mathrm{map}_A"] & \\ A\cdot x & F^m \end{tikzcd}$$

$$\mathrm{map}_A \longleftarrow A, \qquad x \longmapsto A \cdot x$$

$$\underset{m\times n}{A} \underset{n\times 1}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} = \underbrace{A \cdot x}_{\text{matrix product with } x \text{ as a one-column matrix}}$$

$m \times 1$

Let's check that they are inverses:

$$\begin{array}{ccl} F^n & & x \\ {\scriptstyle \mathrm{map}_{A(f)}}\downarrow & & \big\downarrow \\ F^m & & A(f)\cdot x = \begin{pmatrix} | & & | \\ f(e_1) & \cdots & f(e_n) \\ | & & | \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f(e_1)_1 x_1 + \cdots + f(e_n)_1 x_n \\ \vdots \\ f(e_1)_m x_1 + \cdots + f(e_n)_m x_n \end{pmatrix} \end{array}$$

while $f(x) = f(x_1, \dots, x_n) = f(x_1 e_1 + \cdots + x_n e_n)$

$$= x_1 f(e_1) + \cdots + x_n f(e_n)$$

$$= \left(f(e_1)_1 x_1 + \cdots + f(e_n)_1 x_n, \dots, f(e_1)_m x_1 + \cdots + f(e_n)_m x_n \right)$$

so $\mathrm{map}_{A(f)} = f$ ✓

Alt proof Check $\mathrm{map}_{A(f)} e_i = f(e_i)$ for $i = 1, \dots, n$.

Now the other composite:

For $B \in F^{m \times n}$, $A(\mathrm{map}_B) = \begin{pmatrix} | & & | \\ \mathrm{map}_B(e_1) & \cdots & \mathrm{map}_B(e_n) \\ | & & | \end{pmatrix}$

$$= \begin{pmatrix} B \cdot e_1 & \cdots & B \cdot e_n \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ \mathrm{col}_1(B) & \cdots & \mathrm{col}_n(B) \\ | & & | \end{pmatrix}$$

$$= B \quad \checkmark$$

Lemma For $B \in F^{m \times n}$,

$$B \cdot e_i = \mathrm{col}_i(B),$$

the i-th column of B.

Pf Exercise! $\square$

A new view of A_α^β :

$$\begin{tikzcd}
\mathrm{Rep}_\alpha^{-1} x \arrow[rrr, mapsto] & & & f(\mathrm{Rep}_\alpha^{-1} x) \arrow[ddd, mapsto] \\
& V \arrow[r, "f"] \arrow[d, "\mathrm{Rep}_\alpha"'] & W \arrow[d, "\mathrm{Rep}_\beta"] & \\
& F^n \arrow[r, "A_\alpha^\beta(f)\cdot"'] & F^m & \\
x \arrow[uuu, mapsto] \arrow[rr, mapsto] & & A_\alpha^\beta(f)\cdot x \arrow[r, phantom, "="] & \mathrm{Rep}_\beta(f(\mathrm{Rep}_\alpha^{-1} x))
\end{tikzcd}$$

commutes!

i.e. $A_\alpha^\beta(f)\cdot = \mathrm{Rep}_\beta \circ f \circ \mathrm{Rep}_\alpha^{-1}$ and $A_\alpha^\beta(f)$ is the unique matrix with i-th column $\mathrm{Rep}_\beta(f(\mathrm{Rep}_\alpha^{-1}(e_i)))$.

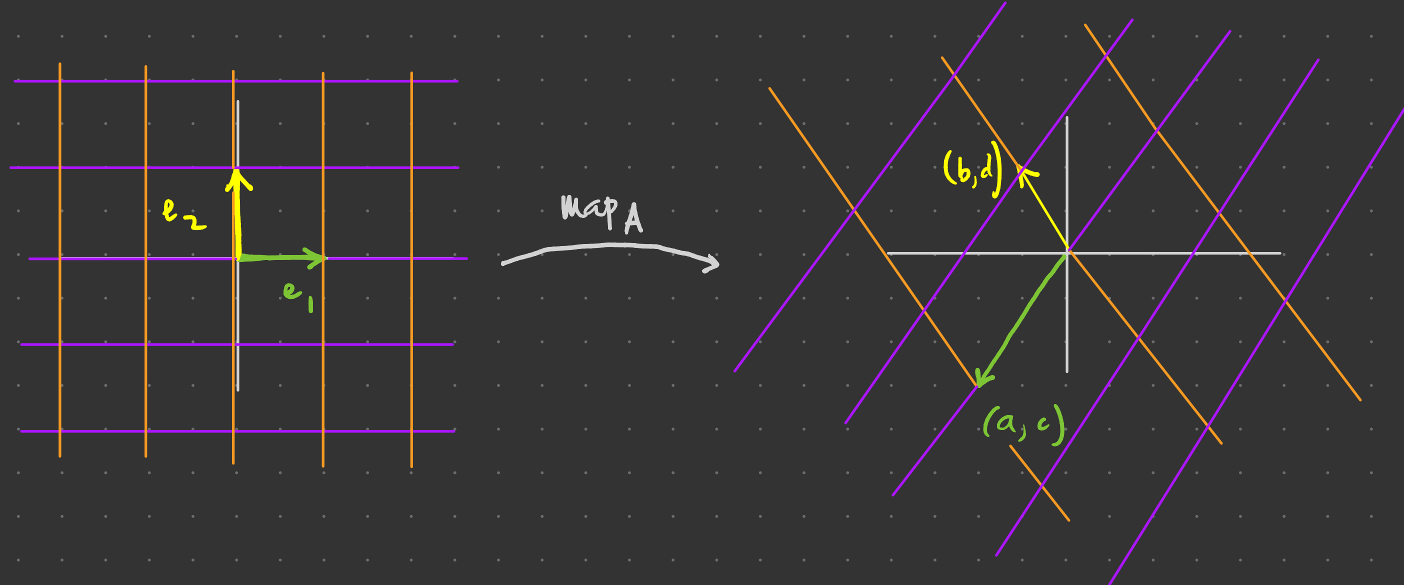
Let's practice with the "standard" setup:

E.g. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \in F^{3\times 2}$

$$\mathrm{map}_A : F^2 \longrightarrow F^3$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3 + 4y \\ 5 + 6y \end{pmatrix}$$

E.g. How does $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$ act on $\mathbb{R}^2$?



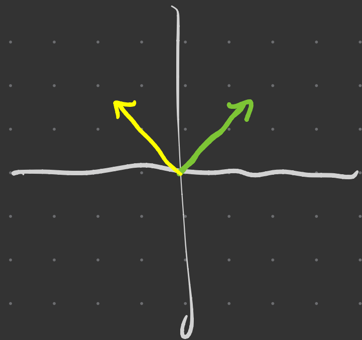
Exercises

(1) Find $A \in \mathbb{R}^{2\times 2}$ that rotates $\mathbb{R}^2$ by $\pi/4$ ccw.

(2) ________ " ________ reflects $\mathbb{R}^2$ over the x-axis.

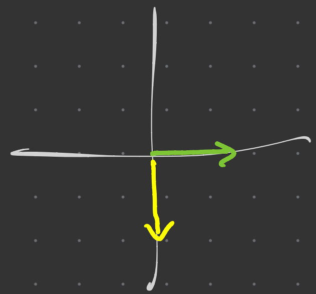
(3) ________ " ________ stretches the x-axis by a factor of 3.

(1) $e_1 \longmapsto \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ $\quad e_2 \longmapsto \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$



so $A = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

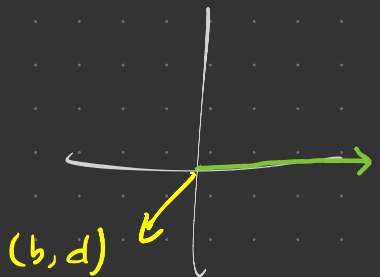
(2) $e_1 \longmapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad e_2 \longmapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix}$



so $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

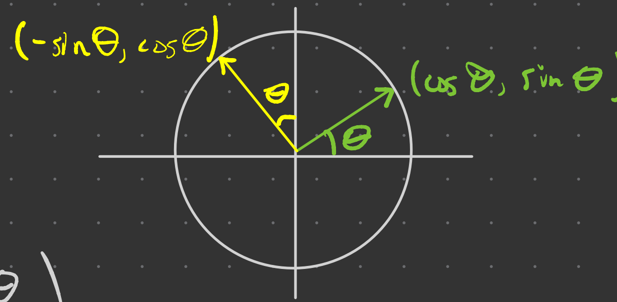
(3) $e_1 \longmapsto \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $\quad e_2 \longmapsto \begin{pmatrix} b \\ d \end{pmatrix}$

so $A = \begin{pmatrix} 3 & b \\ 0 & d \end{pmatrix}$



Arbitrary rotations

Rotate by θ ccw:



$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Complex multiplication

$\mathbb{C} \ni 1 \longleftrightarrow e_1$ $\qquad a, b \in \mathbb{R}$

$\mathbb{C} \ni i = \sqrt{-1} \longleftrightarrow e_2$ $\qquad z = a + bi$ $\qquad z \cdot 1 = z = a + bi \longleftrightarrow ae_1 + be_2$

$z \cdot i = ai - b = -b + ai$

$\longleftrightarrow -be_1 + ae_2$

$$z \cdot \longleftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Image and rank

The image of a linear transformation $f: V \longrightarrow W$ is

$$\operatorname{im}(f) = \{ f(v) \mid v \in V \} \leq W.$$

If V has basis $\{v_1, \dots, v_n\}$, then

$$\operatorname{im}(f) = \operatorname{span}\{ f(v_1), \dots, f(v_n) \}.$$

Thus for $A \in F^{m \times n}$,

$$\begin{aligned} \operatorname{im}(\operatorname{map}_A) &= \operatorname{span}\{ \operatorname{map}_A(e_1), \dots, \operatorname{map}_A(e_n) \} \\ &= \operatorname{span}\{ \operatorname{col}_1(A), \dots, \operatorname{col}_n(A) \} \\ &=: \text{column space of } A \end{aligned}$$

dim = (column) rank of A

So $\operatorname{rank}(A) = \dim \operatorname{im}(f) = \operatorname{rank}(A)$ — good choice of terminology ☺