24. X. 11

Change of basis

Let $\alpha = (v_1, \ldots, v_n)$, $\beta = (w_1, \ldots, w_m)$ be ordered bases of F^n, F^m resp.
If $f : F^n \longrightarrow F^m$ is a linear transformation,

$$\begin{tikzcd} F^n \arrow[r, "f"] \arrow[d, "\mathrm{Rep}_\alpha"'] & F^m \arrow[d, "\mathrm{Rep}_\beta"] \\ F^n \arrow[r, "A_\alpha^\beta(f)"'] & F^m \end{tikzcd}$$

commutes.

conflating $A_\alpha^\beta(f) \in F^{m \times n}$ and its map

Each map has a "matrix name" with respect to the standard ordered bases of F^n, F^m:

$$
\begin{tikzcd}
F^n \arrow[r, "A"] \arrow[d, shift right, "M"'] & F^m \arrow[d, shift right, "N"'] \\
F^n \arrow[r, "B"'] \arrow[u, shift right, "P"'] & F^m \arrow[u, shift right, bend right, "Q"']
\end{tikzcd}
$$

Note $\mathrm{Rep}_\alpha^{-1} : e_i \longmapsto v_i$, $\mathrm{Rep}_\beta^{-1} : e_j \longmapsto w_j$ so

$$
M^{-1} = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} =: P \quad \text{and} \quad N^{-1} = \begin{pmatrix} | & | & & | \\ w_1 & w_2 & \cdots & w_m \\ | & | & & | \end{pmatrix} =: Q.
$$

Thus $\boxed{B = Q^{-1} A P}$.

change of basis formula

E.g. Consider the linear transformation

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$(x,y,z) \longmapsto (x+3y+2z,\ 2y+z)$$

$e_1 \longmapsto (1,0)$

$e_2 \longmapsto (3,2)$

$e_3 \longmapsto (2,1)$

with (std basis) matrix $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix}$.

Choose ordered bases

$$\alpha = \big((1,0,0), (1,1,0), (1,1,1) \big) \text{ of } \mathbb{R}^3$$

$$\beta = \big((0,1), (1,1) \big) \text{ of } \mathbb{R}^2.$$

To find $A^\beta_\alpha(f)$, create

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Compute Q^{-1}: $\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \overset{G-J}{\rightsquigarrow} \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$

so $A^{\beta}_{\alpha}(f) = Q^{-1}AP = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -1 & -2 & -3 \\ 1 & 4 & 6 \end{pmatrix}.$$

$\alpha = \big((1,0,0), (1,1,0), (1,1,1)\big)$ of $\mathbb{R}^3$

$\beta = \big((0,1), (1,1)\big)$ of $\mathbb{R}^2$.

Check: $f(1,0,0) = (1,0) = -(0,1) + (1,1)$

$f(1,1,0) = (4,2) = -2(0,1) + 4(1,1)$

$f(1,1,1) = (6,3) = -3(0,1) + 6(1,1)$ ✓

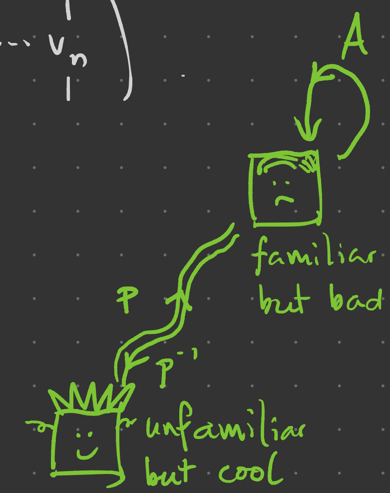
Important special case:

$V = W = F^n$, $\alpha = \beta = (v_1, \dots, v_n)$. Let $P = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$.

If $A \in F^{n\times n}$ and $B = A_\alpha^\alpha(A)$, then

$$\begin{tikzcd} F^n \arrow[r, "A"] \arrow[d, "P^{-1}"'] & F^n \arrow[d, "P^{-1}"] \\ F^n \arrow[r, "B"'] & F^n \end{tikzcd}$$

i.e. $\boxed{B = P^{-1} A P}$.



We say B is formed by <u>conjugating</u> A by P.

<u>E.g.</u> Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $\alpha = ((1,1), (-1,0))$.

Then $P = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and (check) $P^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

Thus $B = P^{-1} A P = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

Huh! $B = A$ in this case!

Check: $$A\begin{pmatrix}1\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

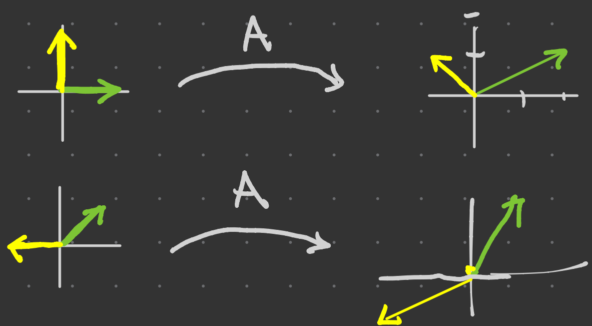
$$A\begin{pmatrix}-1\\0\end{pmatrix} = -\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}-2\\-1\end{pmatrix}$$

Question: What does $A = P^{-1}AP$ mean algebraically? And geometrically?

$$PA = P(P^{-1}AP) = (PP^{-1})AP = AP$$

A and P commute!

$$\begin{pmatrix}2 & -1\\1 & 1\end{pmatrix}$$
$$\shortparallel$$



E.g. Let $\binom{x}{k} = \dfrac{x(x-1)\cdots(x-k+1)}{k!} \in \mathbb{R}[x]$.

For instance, $\binom{x}{0} = 1$

$$\binom{x}{1} = x$$

$$\binom{x}{2} = \frac{x(x-1)}{2} = \frac{1}{2}x^2 - \frac{1}{2}x$$

$$\binom{x}{3} = \frac{x(x-1)(x-2)}{3!} = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x.$$

$\mathbb{R}[x]_{\leq 3}$ as $\mathbb{R}^4$ with

$1 \longleftrightarrow e_1$

$x \longleftrightarrow e_2$

$x^2 \longleftrightarrow e_3$

$x^3 \longleftrightarrow e_4$

Set $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 1/3 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 1/6 \end{pmatrix}$ with columns the $(1, x, x^2, x^3)$ coordinates of $\binom{x}{0}, \binom{x}{1}, \binom{x}{2}, \binom{x}{3}$.

Get
$$\begin{tikzcd} \mathbb{R}[x]_{\le 3} \arrow[r, "\frac{d}{dx}"] \arrow[d, "P^{-1}"'] & \mathbb{R}[x]_{\le 3} \arrow[d, "P^{-1}"] \\ \mathbb{R}[x]_{\le 3} \arrow[r, "A"'] & \mathbb{R}[x]_{\le 3} \end{tikzcd}$$

where A represents differentiation in the $\binom{x}{k}$ basis.

In the x^k basis, $$\frac{d}{dx} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $$A = P^{-1}\frac{d}{dx}P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 1/3 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 1/6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -1/2 & 1/3 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now check:

$$\frac{d}{dx}\binom{x}{0} = 0 \qquad \frac{d}{dx}\binom{x}{2} = \binom{x}{1} - \frac{1}{2}\binom{x}{0}$$

" $$\frac{d}{dx}\left(\frac{1}{2}x^2 - \frac{1}{2}x\right) = x - \frac{1}{2}$$

$$\frac{d}{dx}\binom{x}{1} = \binom{x}{0} \qquad \frac{d}{dx}\binom{x}{3} = \binom{x}{2} - \frac{1}{2}\binom{x}{1} + \frac{1}{3}\binom{x}{0}$$

" $$\frac{d}{dx}\left(\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x\right) = \frac{1}{2}x^2 - x + \frac{1}{3}$$

Fact The entries of P are $\frac{1}{n!}s(n,k)$ for $s(n,k)$ the (signed) Stirling numbers of the first kind.

The entries of P^{-1} are $k!\left\{ {n \atop k} \right\}$ for $\left\{ {n \atop k} \right\}$ the Stirling numbers of the second kind.

Fact A polynomial takes $\mathbb{Z}$ to $\mathbb{Z}$ iff it is an integer linear combo of binomial polynomials $\binom{x}{k}$.

I.e. $a_0 + a_1 x + \cdots + a_n x^n$ takes $\mathbb{Z}$ to $\mathbb{Z}$ iff

$$P^{-1}\begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$$ is a vector with integer coordinates.

(See Tao blog post on Zulip.)