<u>Goals</u>

- $\ker f = \{0\} \Leftrightarrow f$ injective
- isomorphism type is given by dimension

Throughout, V, W F-vs's, $f: V \longrightarrow W$ linear transformation.

Thm (ker $f = 0$ detects injectivity)
A linear transformation f is injective iff $\ker f = 0$.

$\{0_V\}$

now writing 0 for triv subspace of V

Pf ($\Rightarrow$) Since f is linear, $f(0) = 0$, so $0 \in \ker f$. As f is injective, no other vectors map to 0 under f, so $\ker f = 0$.

($\Leftarrow$) Suppose $v, w \in V$ and $f(v) = f(w)$. Then

$$0 = f(v) - f(w) = f(v-w) \quad \text{[linearity]}$$

so $v-w \in \ker f = 0 \Rightarrow v-w=0 \Rightarrow v=w$. Hence f is inj. $\square$

Prop For $S \subseteq V$,

(1) S lin dep $\Rightarrow fS = \{f(s) \mid s \in S\}$ lin dep

(2) S lin ind + f inj $\Rightarrow fS$ lin ind.

Q Why is this a necessary hypothesis?

$s_1 \longmapsto v$

$s_2, \dots, s_n \longmapsto 2v$

$f : V \longrightarrow W$ destroys lin ind

$\underset{\neq 0}{v} \longmapsto 0$

Pf of Prop (1) Suppose $\sum \lambda_i s_i = 0$ for $\lambda_i \in F$, $s_i \in S$. Applying f,

$f(\sum \lambda_i s_i) = \sum \lambda_i f(s_i) = 0$ so f preserves lin dependencies.

(2) Fix $f(s_1), \dots, f(s_n) \in fS$ and suppose

$$\sum \lambda_i f(s_i) = 0 \implies f\left(\sum \lambda_i s_i\right) = 0 \quad \text{[linearity of } f\text{]}$$

$$\implies \sum \lambda_i s_i \in \ker f$$

Since f is injective, $\ker f = 0 \implies \sum \lambda_i s_i = 0$.

Since S lin ind, all $\lambda_i = 0 \implies fS$ lin ind. $\square$

<u>Recall</u> A lin trans'n $f: V \to W$ is an <u>isomorphism</u> when it

- admits a linear inverse $g: W \to V$

or, equivalently,

- is a bijection.

<u>Note</u> Isomorphism is an equivalence rel'n on any set of vector spaces.

$$V \underset{\cong}{\xrightarrow{f}} W \underset{\cong}{\xrightarrow{g}} U \quad \text{then} \quad V \xrightarrow{g \circ f} U \qquad \text{[transitive]}$$

$$f^{-1} \circ g^{-1} = (g \circ f)^{-1}$$

E.g. • $F^{2\times 2} \cong F^4$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (a,b,c,d)$

• Let $F[\![x]\!]$ denote power series with coefficients in F.

Then $F[\![x]\!] \cong F^{\mathbb{N}}$ via $\lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots \longmapsto (i \mapsto \lambda_i)$.

$\quad\;\; \rotatebox{90}{$\subseteq$} \qquad\quad\; \rotatebox{90}{$\subseteq$}$ F-valued sequences ... generating functions!

$\quad F[x] \cong$ sequences that are eventually 0.

Prop A lin trans'n $f: V \to W$ is an isomorphism iff $\underbrace{\ker f = 0}_{\text{inj}}$ and $\underbrace{\operatorname{im} f = W}_{\text{surj}}$. $\square$

Thm If V, W are finite-dimensional F-vs's, then

$$V \cong W \iff \dim V = \dim W.$$

$(\Rightarrow)$

<u>Pf</u> Since $\cong$ is symmetric, it suffices to prove that

$$\dim V = n \implies V \cong F^n.$$

Choose an ordered basis $B = (b_1, \dots, b_n)$ of V. Then

$$\mathrm{Rep}_B : V \xrightarrow{\cong} F^n, \quad b_i \longmapsto e_i .$$

$(\Leftarrow)$ Isomorphisms $f: V \to W$ preserve bases: B basis of V. Then fB is lin ind (prop) and if $w \in W$ $\exists!\, v \in V$ s.t. $f(v) = w$ and $f(\sum \lambda_i b_i) = \sum \lambda_i f(b_i)$ so $w \in \operatorname{span} fB$ $\square$

<u>Note</u> Specifying an iso $V \xrightarrow{\cong} F^n$ is equivalent to choosing an ordered basis of V b/c the inverse $g: F^n \longrightarrow V$ gives $g(e_1), \dots, g(e_n)$ as a basis of V.

<u>Week 3 #6</u> Claim $S = \{(a,b), (c,d)\} \subseteq F^2$ lin ind

iff $ad - bc \neq 0$.

★ Given $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $\Delta = \Delta(A) = ad - bc$.

(a,b), (c,d) are lin ind iff

$$\operatorname{rref}\left(\begin{array}{cc|c} a & c & 0 \\ b & d & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

Note (1) elementary row ops take $A := \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ to $\operatorname{rref}(A)$.

(2) observe what each op'n does to Δ:

$\underline{A}$	$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$	$\begin{pmatrix} b & d \\ a & c \end{pmatrix}$	$\begin{pmatrix} \lambda a & \lambda c \\ b & d \end{pmatrix}$	$\begin{pmatrix} a & c \\ \lambda b & \lambda d \end{pmatrix}$	$\begin{pmatrix} a & c \\ b+\lambda a & d+\lambda c \end{pmatrix}$
$\underline{\Delta}$	$ad-bc$	$bc-ad$ $= -1\cdot(ad-bc)$	$\lambda(ad-bc)$	$\lambda(ad-bc)$	$ad+\cancel{\lambda ac}$ $-bc-\cancel{\lambda ac}$ $= ad-bc$

Thus the effect of a row op on Δ is to scale by a nonzero λ

Since $\mathrm{rref}(A)$ is given by a sequence of row ops,

$$\Delta(A) = \lambda \cdot \Delta(\mathrm{rref}(A))$$

so S lin ind iff $\text{rref}(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Delta = 1 \implies \Delta(A) = \lambda \cdot 1, \quad \lambda \neq 0$$
$$\neq 0$$