

- Goals
- Define image & kernel
 - Rank-nullity theorem

Defn Given a linear transformation $f: V \rightarrow W$, the image of f is $\text{im}(f) := fV = \{f(v) \mid v \in V\}$.

Prop $\text{im}(f) \leq W$

Pf (0) We showed $f(0) = 0$ for f linear, so $0 \in \text{im } f$.

(1) Suppose $w_1, w_2 \in \text{im } f$. By defn $\exists v_1, v_2 \in V$ s.t.
 $f(v_1) = w_1, f(v_2) = w_2$. Then $w_1 + w_2 = f(v_1) + f(v_2)$
 $= f(v_1 + v_2)$ by linearity so $w_1 + w_2 \in \text{im } f$.

(2) If $w = f(v) \in \text{im } f$, then $\forall \lambda \in F, \lambda w = \lambda f(v) = f(\lambda v) \in \text{im } f$. $\square$

Defn The rank of a linear trans'n $f: V \rightarrow W$ is
$$\text{rank}(f) := \dim \text{im}(f).$$

E.g. Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and linear extension:
$$\begin{aligned} e_1 &\longmapsto (2, 1, 0) \\ e_2 &\longmapsto (0, -1, 1) \end{aligned}$$

$$f(x, y) = f(xe_1 + ye_2)$$

$$= x(2, 1, 0) + y(0, -1, 1) = (2x, x-y, y)$$

Then $\text{im}(f) = \text{span}\{(2, 1, 0), (0, -1, 1)\}$ and $\text{rank}(f) = 2$ (why?)

Rmk If $f: V \rightarrow W$ linear, $\{b_1, \dots, b_n\}$ basis of V , then

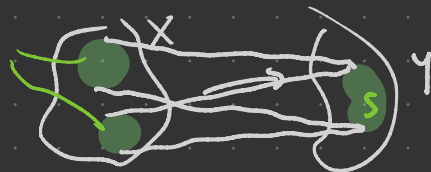
$$\text{im}(f) = \text{span}\{f(b_1), \dots, f(b_n)\}$$

If $f(b_1), \dots, f(b_n)$ lin dependent, then $\text{rank}(f) < n$.

Recall $f: X \rightarrow Y$, $S \subseteq Y$ then the preimage of S is

$$f^{-1}S := \{x \in X \mid f(x) \in S\}.$$

$$f^{-1}: 2^Y \rightarrow 2^X$$



Prop For $f: V \rightarrow W$ linear and $U \subseteq W$, $f^{-1}U \subseteq V$.

Pf (1) Note $0 \in U$ b/c U subspace and $f(0) = 0$ so $0 \in f^{-1}U$.

(1) Suppose $v_1, v_2 \in f^{-1}U$ so $f(v_1) = u_1, f(v_2) = u_2 \in U$.

Thus $f(v_1 + v_2) = f(v_1) + f(v_2) = u_1 + u_2 \in U$ b/c U is a subspace.

Hence $v_1 + v_2 \in f^{-1}U$.

(2) Exc. $\square$

Defn For $f: V \rightarrow W$ linear, the kernel (or nullspace) of f is

$$\ker(f) := f^{-1}\{0\} = \{v \in V \mid f(v) = 0\}$$

Note Since $\{0\} \subseteq W$, $\ker f \subseteq V$.

Defn The nullity of f is $\dim \ker f$.

Problem Determine the kernel and nullity of the linear trans'n

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$e_1 \mapsto (2, 1, 0)$$

$$e_2 \mapsto (0, -1, 1)$$

$$(x, y) \in \ker f$$

$$\Leftrightarrow f(x, y) = 0 \Leftrightarrow x(2, 1, 0) + y(0, -1, 1) = (0, 0, 0)$$

$$\Leftrightarrow x = y = 0 \quad \text{So } \ker(f) = \{0\} \subseteq \mathbb{R}^2$$

E.g. Consider the linear trans'n

$$\frac{d}{dx} : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 1}$$

$$a+bx+cx^2 \longmapsto b+2cx$$

$$\text{Then } \ker\left(\frac{d}{dx}\right) = \left\{ a+bx+cx^2 \mid b+2cx=0 \right\}$$

with domain
 $\mathbb{R}[x]_{\leq 2}$

$$= \left\{ a \in \mathbb{R}[x]_{\leq 2} \mid a \in \mathbb{R} \right\}$$

$$= \text{span}\{1\}$$

and the nullity of $\frac{d}{dx}$ is 1.

The image of $\frac{d}{dx}$ is

apply $\frac{d}{dx}$ to basis of $\mathbb{R}[x]_{\leq 2}$

$$\text{im } \frac{d}{dx} = \text{span} \left\{ \frac{d}{dx} 1, \frac{d}{dx} x, \frac{d}{dx} x^2 \right\}$$

$$= \text{span} \{0, 1, 2x\}$$

$$= \mathbb{R}[x]_{\leq 1}$$

$$\text{so } \text{rank } \frac{d}{dx} = 2.$$

Note $3 = \dim \mathbb{R}[x]_{\leq 2} = \text{rank} \left(\frac{d}{dx} \right) + \text{null} \left(\frac{d}{dx} \right)$

— this is generic.

Thm (rank-nullity)

Suppose $f: V \rightarrow W$ linear, V is finite diml. Then

$$\dim V = \text{rank}(f) + \text{null}(f).$$

... { Domain "splits" into kernel and the image.

Pf Suppose $\dim(f) = k$ and $\ker(f)$ has basis $\{v_1, \dots, v_k\}$.

We can complete this to a basis $B = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V .

It suffices to show $\{f(v_{k+1}), \dots, f(v_n)\}$ is a basis of $\operatorname{im}(f)$.

$$\begin{aligned} \text{Now } \operatorname{im}(f) &= \underbrace{fB}_{\text{span}} = \operatorname{span}\{f(v_1), \dots, f(v_n)\} \\ &= \operatorname{span}\{0, \dots, 0, f(v_{k+1}), \dots, f(v_n)\} \end{aligned}$$

so $\{f(v_{k+1}), \dots, f(v_n)\}$ generates $\operatorname{im}(f)$.

For linear ind, suppose

$$\lambda_{k+1} f(v_{k+1}) + \dots + \lambda_n f(v_n) = 0.$$

Then $f(\lambda_{k+1}v_{k+1} + \dots + \lambda_nv_n) = 0$

$\Rightarrow \lambda_{k+1}v_{k+1} + \dots + \lambda_nv_n \in \ker f$ ← has basis $\{v_1, \dots, v_k\}$

$\Rightarrow \exists \lambda_1, \dots, \lambda_k \in F$ s.t.

$$\lambda_1v_1 + \dots + \lambda_kv_k = \lambda_{k+1}v_{k+1} + \dots + \lambda_nv_n$$

But then $\lambda_1v_1 + \dots + \lambda_kv_k - \lambda_{k+1}v_{k+1} - \dots - \lambda_nv_n = 0$

and by lin ind of $B = \{v_1, \dots, v_n\}$, get $\lambda_i = 0$ for all i .

In particular, $f(v_{k+1}), \dots, f(v_n)$ are lin ind. □

Suppose $S = \{v_1, \dots, v_k\}$ lin ind in V .

Basis ext'n algorithm:

- ① If $\text{span } S = V$, done.
- ② O/w take $v_{k+1} \in V \setminus \text{span } S$.
- ③ Add v_{k+1} to S ————— $S \cup \{v_{k+1}\}$ is lin ind.
- ④ Re-index and go back to ①.

If V is fin diml, this will terminate after
 $\dim V - k$ steps.

E.g. $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ extend linearly

$$\begin{aligned} e_1 &\longmapsto (1, 1, 1) \\ e_2 &\longmapsto (-2, 0, 3) \\ e_3 &\longmapsto (1, 3, 6) \end{aligned}$$

Find $\ker f = \{(x, y, z) \mid f(x, y, z) = 0\}$

$$= \{(x, y, z) \mid x(1, 1, 1) + y(-2, 0, 3) + z(1, 3, 6) = (0, 0, 0)\}$$

So need to solve

$$x - 2y + z = 0$$

$$x + 3z = 0$$

$$x + 3y + 6z = 0$$

$$A = \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 3 & 6 & 0 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 5 & 5 & 0 \end{array} \right)$$

$\longrightarrow$

$\dots$

$$\text{rank} = 2$$

$$\Rightarrow \text{nullity} = 1.$$