Goal Invent matrix multiplication

Recall A linear transformation determines and is determined by its action on a basis.

Encode linear trans'ns $V \xrightarrow{f} W$ by

① Choose ordered bases $\alpha = (v_1, \dots, v_n)$, $\beta = (w_1, \dots, w_m)$ of V, W, resp.

② Record $\mathrm{Rep}_\beta f(v_1), \mathrm{Rep}_\beta f(v_2), \dots, \mathrm{Rep}_\beta f(v_n) \in F^m$ as the columns of an $m \times n$ matrix

$$A_\alpha^\beta(f) = \begin{pmatrix} | & & | \\ \mathrm{Rep}_\beta f(v_1) & \cdots & \mathrm{Rep}_\beta f(v_n) \\ | & & | \end{pmatrix} \in F^{m \times n}$$

This is the matrix representing f relative to the ordered bases α, β.

This produces a linear transformation

$$A_\alpha^\beta : \mathrm{Hom}(V, W) \longrightarrow F^{m\times n}$$
$$f \longmapsto A_\alpha^\beta(f).$$

(check! $A_\alpha^\beta(f+g) = A_\alpha^\beta(f) + A_\alpha^\beta(g)$,

$A_\alpha^\beta(\lambda f) = \lambda A_\alpha^\beta(f)$)

Q Why surjective?

A Given $A = (a_{ij}) \in F^{m\times n}$, take $f : V \to W$ linear

$$v_j \longmapsto \underbrace{\sum_i a_{ij} w_i}_{\text{vector with Rep}_\beta \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}}$$

Then $A_\alpha^\beta(f) = A$.

Q Why injective?

A $\ker(A_\alpha^\beta) = \{ f \in \mathrm{Hom}(V,W) \mid A_\alpha^\beta(f) = (0) \}$

$= \{f \mid f(v_j) = 0 \ \forall j\} = \{0 : V \to W\}.$

Upshot $\boxed{A_\alpha^\beta : \mathrm{Hom}(V, W) \cong F^{m \times n}}.$

But wait — there's more!

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & V \arrow[r, "g"] & W \end{tikzcd}$$

Suppose U has ord'd basis $\alpha = (u_1, \dots, u_n)$

V —"— $\beta = (v_1, \dots, v_m)$

W —"— $\gamma = (w_1, \dots, w_l)$

Get matrices $A_\alpha^\beta(f) \in F^{m \times n}$

$A_\beta^\gamma(g) \in F^{l \times m}$

$A_\alpha^\gamma(g \circ f) \in F^{l \times n}$

$$\begin{tikzcd} \mathrm{Hom}(V,W) \times \mathrm{Hom}(U,V) \arrow[r, "\circ"] \arrow[d, "\cong"'] & \mathrm{Hom}(U,W) \arrow[d, "\cong"] \\ F^{l \times m} \times F^{m \times n} \arrow[r, "\cdot"] & F^{l \times n} \end{tikzcd}$$

Goal Define a function

$$F^{l\times m} \times F^{m\times n} \longrightarrow F^{l\times n}$$

$$(A, B) \longmapsto A\cdot B$$

such that $A_\alpha^\gamma(g\circ f) = A_\beta^\gamma(g)\cdot A_\alpha^\beta(f)$

🤞 The function won't depend on $\alpha, \beta, \gamma, f, g$, but rather only on matrix entries.

Let's compute $(g\circ f)(u_j) = g(f(u_j))$ k-th entry of j-th col of $[A_\alpha^\beta(f) = (b_{kj})]$

$$= g\left(\sum_k b_{kj} v_k\right)$$

$$= \sum_k b_{kj}\, g(v_k) \qquad \text{[linearity]}$$

$$= \sum_k b_{kj} \left(\sum_i a_{ik} w_i \right)$$

i-th entry of k-th col of $[A_\beta^\gamma(g) = (a_{ik})]$

$$= \sum_i \underbrace{\left(\sum_k a_{ik} b_{kj} \right)}_{(i,j)\text{ entry of } A_\alpha^\gamma(g\cdot f)} w_i$$

Thus for arbitrary $A = (a_{ij}) \in F^{l \times m}$, $B = (b_{ij}) \in F^{m \times n}$,

we must define $A \cdot B \in F^{l \times n}$ to be the matrix (c_{ij}) with

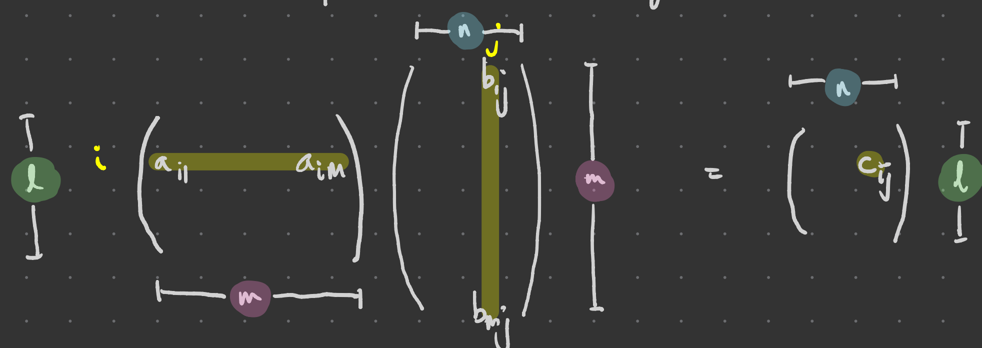
$$\boxed{c_{ij} = \sum_k a_{ik} b_{kj}}.$$

With this operation, we have:

Thm $A^{\gamma}_{\alpha}(g \circ f) = A^{\gamma}_{\beta}(g) \cdot A^{\beta}_{\alpha}(f)$

i.e. matrix multiplication encodes composition of linear transformations!

Matrix multiplication visually:



$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

E.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 1 + 2(-1) + 3\cdot 2 & 1(-1) + 2\cdot 1 + 3\cdot 0 \\ 4\cdot 1 + 5(-1) + 6\cdot 2 & 4(-1) + 5\cdot 1 + 6\cdot 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 1 \\ 11 & 1 \end{pmatrix}$$

often called the dot product of $(a_{i1}, \dots, a_{im})$ and $(b_{1j}, \dots, b_{mj})$.

Alternative visualization:

$$\begin{array}{cc} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 2 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} & \begin{pmatrix} 5 & 1 \\ 11 & 1 \end{pmatrix} \end{array}$$

Now consider $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3$ $\qquad$ $\mathbb{R}^3 \xrightarrow{g} \mathbb{R}^2$ $\qquad$ lin trans'ns.

$$e_1 \longmapsto (1,-1,2) \qquad e_1 \longmapsto (1,4)$$
$$e_2 \longmapsto (-1,1,0) \qquad e_2 \longmapsto (2,5)$$
$$\qquad\qquad\qquad\qquad e_3 \longmapsto (3,6)$$

Then $\quad A(f) = A^{\mathcal{E}_3}_{\mathcal{E}_2}(f) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 2 & 0 \end{pmatrix}$

$$A(g) = A^{\mathcal{E}_2}_{\mathcal{E}_3}(g) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

<u>Notation</u> Write

$\mathcal{E}_n := (e_1, \dots, e_n)$

for the standard ord basis of $\mathbb{R}^n$

and the previous computation implies

$$A(g \circ f) = A^{\mathcal{E}_2}_{\mathcal{E}_2}(g \circ f) = A(g) \cdot A(f) = \begin{pmatrix} 5 & 1 \\ 11 & 1 \end{pmatrix}.$$

Thus $g \circ f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the unique linear trans'n such that $e_1 \longmapsto (5,11)$, $e_2 \longmapsto (1,1)$.

<u>E.g.</u> Consider $V = \operatorname{span}\{\cos, \sin\} \leq \mathbb{R}^{\mathbb{R}}$ with ordered basis $\underbrace{(\cos, \sin)}_{\alpha :=}$.

$$\frac{d}{dx} : V \longrightarrow V \qquad \text{so} \qquad A\left(\frac{d}{dx}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$\cos \longmapsto -\sin$$

$$\sin \longmapsto \cos$$

$$A^{\alpha}_{\alpha}\left(\frac{d}{dx}\right)$$

$\operatorname{Rep}_\alpha(-\sin) = (0,-1)$

$\operatorname{Rep}_\alpha(\cos) = (1,0)$

Note $\dfrac{d^2}{dx^2} = \dfrac{d}{dx} \circ \dfrac{d}{dx}$ so

$$A\left(\frac{d^2}{dx^2}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

And indeed, $\frac{d^2}{dx^2} : \cos \longmapsto -\cos, \ \sin \longmapsto -\sin.$

Now $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to $id : V \longrightarrow V$.

so $\frac{d^4}{dx^4} = id$ on V.