

- Goals
- Define row & column spaces of a matrix
 - Use Gauss-Jordan reduction to compute bases for both
 - Prove that dimensions of row & column spaces are equal
 $\leadsto$ def'n of rank of a matrix

Defn • For $A \in F^{m \times n}$, the row space of A is the span of the rows of A in F^n ; the column space of A is the span of the columns of A in F^m .

- The row rank of A is the dimension of its row space;
 the column rank " " " " column space.

Recall The elementary row operations:

- are linear combinations/reorderings of rows
- are reversible.

Thus $\text{row space of } A = \text{row space of } \text{rref}(A)$



Thm The nonzero rows of $\text{rref}(A)$ form a basis of the row space of A .
Pf Shape! $\square$

Eg. $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 2/3 & -4 \\ 0 & 1 & -1/3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

these are a basis
for row space of A

The row rank of A is 2.

Prop Let $A = (c_1 \ c_2 \ \dots \ c_n) \in F^{m \times n}$ with $c_i \in F^m$ (as col vectors).

Let $\tilde{A}$ be any matrix formed by applying row ops to A , and let $\tilde{c}_1, \dots, \tilde{c}_n$ be the columns of $\tilde{A}$. Then for $\lambda_1, \dots, \lambda_n \in F$,

$$\sum_{i=1}^n \lambda_i c_i = 0 \quad \textcircled{1} \iff \sum_{i=1}^n \lambda_i \tilde{c}_i = 0. \quad \textcircled{2}$$

Pf We have $\textcircled{1} \iff \lambda_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \dots + \lambda_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = 0$ and sol'n's

to this are systems $\underset{c_1}{a_{11}}\lambda_1 + \underset{c_n}{a_{1n}}\lambda_n + \dots = 0$

$$\vdots$$
$$a_{m1}\lambda_1 + a_{m2}\lambda_2 + \dots + a_{mn}\lambda_n = 0.$$

Row operations don't change sol'n sets, so the result follows. $\square$

Cor Let $E = \text{rref}(A)$ with pivot column indices $j_1, \dots, j_r$. Then the columns of A indexed by $j_1, \dots, j_r$ form a basis of the column space of A .



these cols of A — not of $\text{rref}(A)$

Pf For simplicity, assume $j_1 = 1, \dots, j_r = r$. Then E looks like

$$\begin{pmatrix} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$E_1 \quad E_2 \quad \dots \quad E_n$

in the $m=5, n=6, r=3$ case.

Let $E_1, \dots, E_n$ denote cols of E .

$A_1, \dots, A_n$ cols of A .

WTS $A_1, \dots, A_r$ are lin ind and generate col space.

Lin ind : Suppose $\lambda_1 A_1 + \dots + \lambda_r A_r = 0 \xRightarrow{\text{prop}} \lambda_1 E_1 + \dots + \lambda_r E_r = 0$.

But $E_1, \dots, E_r$ are lin ind, so $\lambda_1 = \dots = \lambda_r = 0$.

Generates To show $\text{span}\{A_1, \dots, A_r\} = \text{col space of } A$, it suffice to show $A_j \in \text{span}\{A_1, \dots, A_r\}$ for $j > r$.

Since $E_1, \dots, E_r$ span the col space of E (why? ✓), know

$$\exists \lambda_1, \dots, \lambda_r \in F \text{ s.t. } \lambda_1 E_1 + \dots + \lambda_r E_r = E_j$$

$$\Leftrightarrow \lambda_1 E_1 + \dots + \lambda_r E_r - E_j = 0$$

$$\Leftrightarrow \lambda_1 A_1 + \dots + \lambda_r A_r - A_j = 0$$

$$\xRightarrow{\text{prop}} \lambda_1 A_1 + \dots + \lambda_r A_r = A_j. \quad \square$$

Eg. $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 2/3 & -4 \\ 0 & 1 & -1/3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so $\left\{ \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} \right\}$ is a basis for the column space of A ,

and its column rank is 2. 🤔 Same as row rank?

Thm Row rank of A = column rank of A .

Pf let $E = \text{rref}(A)$. The # nonzero rows of E = # pivot columns of E . $\square$

Defn The rank of A , denoted $\text{rank}(A)$, is its row (or column) rank.

Rank is a unique solution detector:

To compute sol's of a ^{homogeneous} linear system

$$a_{11}x_1 + \dots + a_{1n}x_n = 0$$

$\vdots$

$$a_{m1}x_1 + \dots + a_{mn}x_n = 0$$

Let $A = (a_{ij}) \in F^{m \times n}$ and compute $\text{rref}(A)$.

The #free variables = #non-pivot columns
= $n - \text{rank}(A)$.

$$\boxed{\exists! \text{ sol'n} \iff \text{rank}(A) = n}$$

Note Set of sol's

is always a subspace
of F^n with

dimension = #free
vars.

For a non-homogeneous system

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

$\vdots$

$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

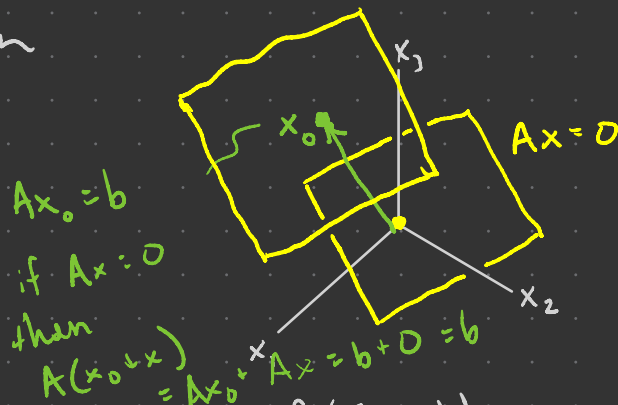
set $A = (a_{ij})$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ and compute $\text{rref}((A|b))$.

If the system is consistent, then every sol'n is of the form

(particular sol'n) + (sol'n of $Ax = 0$).

So if the system is consistent, there is again

$$\text{unique sol'n} \iff \text{rank}(A) = n$$



E.g. How many sol'ns does

$$2x = 2 \quad \text{have?}$$

$$3y + 2z = 12$$

$$y - z = -1$$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 12 \\ -1 \end{pmatrix}$$

$$\frac{5}{3} \lambda = 2$$

$$\lambda = \frac{6}{5}$$

G-J red'n for A:

$$A|b \xrightarrow{r_3 \rightarrow r_3 - \frac{1}{3}r_2} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 3 & 2 & 12 \\ 0 & 0 & -5/3 & -5 \end{array} \right) \xrightarrow{r_2 \rightarrow r_2 + \frac{6}{5}r_3} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & -5/3 & -5 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$\Rightarrow$ a unique soln to $Ax = b$ iff it's consistent:

$$\text{rank}(A) = 3$$

I, consistent and sol'n set is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$.

Note col space of A has basis $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$.