

$B_n = \{w_1, \dots, w_n\}$ is a basis of V

in

C

In fact, $B_n = C$ ^{because otherwise} b/c o/w $w_{n+1} \in \text{span } B_n$

$\Rightarrow C$ lin dep. ∞ $\square$

Cor dimension of fin dim vs's is well-defined

Cor If V is a fin dim vs, $S \subseteq V$ lin ind, then we may extend S to a basis of V by adding some $\dim V - |S|$ elements.

Pf Apply the "basis production algorithm" from the theorem's proof. $\square$

Cor If V fin dim vs and $T \subseteq V$ generates V , then some subset $S \subseteq T$ is a basis of V . $\square$

Cor If $S \subseteq V$ and $|S| = n = \dim V < \infty$, then S is lin ind
iff $\text{span } S = V$. $\square$

$$e_i = (0, \dots, 0, \overset{i\text{-th}}{1}, 0, \dots, 0)$$

E.g. (1) F^n has basis $\{e_1, \dots, e_n\}$ so $\dim F^n = n$.

(2) Let $S = \{(1, 0, 0), (1, 2, 0), (1, 2, 3)\} \subseteq \mathbb{R}^3$. Since

(a) $(1, 2, 0) \notin \text{span}\{(1, 0, 0)\}$

(b) $(1, 2, 3) \notin \text{span}\{(1, 0, 0), (1, 2, 0)\}$

(c) $\dim \mathbb{R}^3 = 3 = |S|$

know S is a basis of $\mathbb{R}^3$.

(3) $\dim F[x]_{\leq n} = n+1$ basis $\{1, x, x^2, \dots, x^n\}$
polynomials of deg $\leq n$

$$(4) \dim F^{m \times n} = mn.$$

Problem Determine $\dim F[x, y]_{\leq n}$ where

- $F[x, y]$ = 2-var polynomials over F ,
- $\deg \sum \lambda_{ij} x^i y^j = \max \{ i+j \mid \lambda_{ij} \neq 0 \}$
- $F[x, y]_{\leq n} = \{ f \in F[x, y] \mid \deg f \leq n \}.$

Condorcet's Voting Paradox

Candidates A, B, C

29 voters

Preferences

voters

A > B > C

5

A > C > B

4

B > A > C

2

B > C > A

8

C > A > B

8

C > B > A

2

model assumption: every voter can linearly order preferences

⇒

$$A > B : 17 - 12 = 5$$

$$B > C : 15 - 14 = 1$$

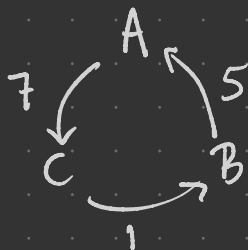
$$C > A : 18 - 11 = 7$$

A > B > C > A — Condorcet cycle

Task Teams A, B, C schedule sequential head-to-head votes so your candidate wins — become a dictator through bureaucracy!

C: first do A vs B — A wins
then A vs C — C wins

We can visualize the total binary preferences as



This is called a Condorcet cycle and it leads to a voting paradox.

Goal Use linear algebra to understand how/when such cycles exist.

Marie Jean Antoine
Nicolas de Caritat,
Marquis of Condorcet
1743-1794 CE



$$\text{Set } V = \left\{ \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ c \quad \quad b \\ \rightarrow B \end{array} \mid a, b, c \in \mathbb{R} \right\} \cong \mathbb{R}^3$$

(a, b, c)

An $A > B > C$ voter corresponds to $\begin{array}{c} A \\ \swarrow \quad \nwarrow \\ -1 \quad \quad 1 \\ \rightarrow B \\ \downarrow \\ C \end{array}$, etc

In the above example, the election is the vector

$$5 \cdot \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ -1 \quad \quad 1 \\ \rightarrow B \\ \downarrow \\ C \end{array} + 4 \cdot \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ -1 \quad \quad 1 \\ \rightarrow B \\ \downarrow \\ C \end{array} + \dots + 2 \cdot \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ 1 \quad \quad -1 \\ \rightarrow B \\ \downarrow \\ C \end{array}$$

(A > B > C) (A > C > B) (C > B > A)

Defn Call a vector in $\mathbb{R}^3$ purely cyclic when it is of the form

$(\lambda, \lambda, \lambda) = \lambda(1, 1, 1)$ for some $\lambda \in \mathbb{R}$. Let

$C = \text{span} \left\{ \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ 1 \quad \quad -1 \\ \rightarrow B \\ \downarrow \\ C \end{array} \right\}$ be the subspace of purely cyclic vectors.

To have no cyclic component is to be perpendicular to C :

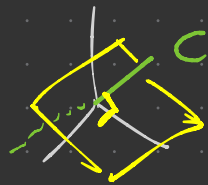
$$(a, b, c) \perp (x, y, z) \iff \underbrace{ax + by + cz}_{(a, b, c) \cdot (x, y, z)} = 0$$

$(a, b, c) \cdot (x, y, z)$ — more on this
when we study inner product spaces

$$\text{So } C^\perp = \{ (a, b, c) \in \mathbb{R}^3 \mid ak + bk + ck = 0 \quad \forall k \in \mathbb{R} \}$$

$$= \{ (a, b, c) \in \mathbb{R}^3 \mid a + b + c = 0 \}$$

$$= \{ b(-1, 1, 0) + c(-1, 0, 1) \mid b, c \in \mathbb{R} \}$$



Get an ordered basis

$$B = \left(\underbrace{(1, 1, 1)}_C, \underbrace{(-1, 1, 0), (-1, 0, 1)}_{C^\perp} \right) \text{ of } \mathbb{R}^3$$

If $\text{Rep}_B(a, b, c) = (x, y, z)$, call x the cyclic component, y, z the non-cyclic components.

$$\begin{array}{c} A \\ \swarrow \quad \searrow \\ -1 \quad 1 \\ \downarrow \quad \uparrow \\ B \end{array} \quad A > B > C$$

E.g. $\text{Rep}_B(1, 1, -1) = (x, y, z) \iff (1, 1, -1) = x(1, 1, 1) + y(-1, 1, 0) + z(-1, 0, 1)$

$$\iff \begin{cases} x - y - z = 1 \\ x + y = 1 \\ x + z = -1 \end{cases} \iff \left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{array} \right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & -4/3 \end{array} \right)$$

so $\text{Rep}_B(1, 1, -1) = \left(\frac{1}{3}, \frac{2}{3}, -\frac{4}{3} \right)$



Rational preference

$A > B > C$ has a cyclic component!

Similarly, $\text{Rep}_B(C > B > A) = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{4}{3} \right)$.

Defn The sign of the cyclic component is the spin of the vector.

Positive Spin

$$\begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{1} \quad B \\ \downarrow \\ A > B > C \end{array} = \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{1/3} \quad B \\ \downarrow \\ A > B > C \end{array} + \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{2/3} \quad B \\ \downarrow \\ A > B > C \end{array}$$

$$\begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{1} \quad B \\ \downarrow \\ B > C > A \end{array} = \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{1/3} \quad B \\ \downarrow \\ B > C > A \end{array} + \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{2/3} \quad B \\ \downarrow \\ B > C > A \end{array}$$

$$\begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{-1} \quad B \\ \downarrow \\ C > A > B \end{array} = \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{1/3} \quad B \\ \downarrow \\ C > A > B \end{array} + \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{-4/3} \quad B \\ \downarrow \\ C > A > B \end{array}$$

cyclic sum of non-cyclic

Negative Spin

$$\begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{-1} \quad B \\ \downarrow \\ C > B > A \end{array} = \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{-1/3} \quad B \\ \downarrow \\ C > B > A \end{array} + \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{-2/3} \quad B \\ \downarrow \\ C > B > A \end{array}$$

$$\begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{-1} \quad B \\ \downarrow \\ A > C > B \end{array} = \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{-1/3} \quad B \\ \downarrow \\ A > C > B \end{array} + \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{-2/3} \quad B \\ \downarrow \\ A > C > B \end{array}$$

$$\begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{1} \quad B \\ \downarrow \\ B > A > C \end{array} = \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{-1/3} \quad B \\ \downarrow \\ B > A > C \end{array} + \begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{4/3} \quad B \\ \downarrow \\ B > A > C \end{array}$$

Contribution

$$\begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{a} \quad B \\ \downarrow \\ -a \end{array}$$

$$\begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{b} \quad B \\ \downarrow \\ b \end{array}$$

$$\begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{-c} \quad B \\ \downarrow \\ c \end{array}$$

The election results in a Condorcet cycle when all three sides of $\begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{a+b+c} \quad B \\ \downarrow \\ a+b-c \end{array}$ have the same sign

$$\begin{array}{c} A \\ \swarrow \quad \nwarrow \\ C \quad \xrightarrow{a+b+c} \quad B \\ \downarrow \\ a+b-c \end{array}$$

If all positive,

$$\begin{array}{rcl} -a + b + c > 0 & \xrightarrow{+} & 2c > 0 \Rightarrow c > 0 \\ a - b + c > 0 & \xrightarrow{+} & 2b > 0 \Rightarrow b > 0 \\ a + b - c > 0 & \xrightarrow{+} & 2a > 0 \Rightarrow a > 0 \end{array}$$

Similarly, if all negative, $a, b, c < 0$.

Thm If there is a Condorcet cycle, then $a, b, c > 0$ or $a, b, c < 0$.

Is the converse true?

$$\begin{array}{lll} a = 15 & -a + b + c = -13 \\ b = 1 & a - b + c = 15 & \text{so not a Cond-} \\ c = 1 & a + b - c = 15 & \text{orcet cycle} \end{array}$$

Fact As the # of voters $\rightarrow \infty$,

$$P(\text{Condorcet cycle}) \rightarrow \frac{\arcsin \frac{\sqrt{6}}{9}}{\pi} \approx 0.0877$$