

- Goals
- define linear transformations
 - proving linearity
 - determined by action on a basis
 - $\text{Hom}(V, W)$ as a vector space

or only diff'l
or diff'l fns

Sets : Functions $\therefore$ open subsets of $\mathbb{R}$: continuous functions

$\therefore$ vector spaces : linear transformations

Defn Fix vector spaces V, W over F . A linear transformation

$L: V \rightarrow W$ is a function s.t. $\forall u, v \in V, \lambda \in F$, (or homomorphism)

$$L(u+v) = L(u) + L(v) \quad \& \quad L(\lambda u) = \lambda L(u).$$

Equivalently, $L(u + \lambda v) = L(u) + \lambda L(v)$.



Huffman's "linear transformations" have the same domain & codomain: $L: V \rightarrow V$. These are typically called linear endomorphisms.

E.g. $L: V \rightarrow W$
 $\quad \quad \quad \downarrow \quad \quad \quad \mapsto 0$

E.g. $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$(x, y, z) \mapsto (2x + 3y, x + y - 3z)$ is linear.

Exc What assignment on std basis of $\mathbb{R}^3$ determines this lin trans?

If let $(x, y, z), (x', y', z') \in \mathbb{R}^3, \lambda \in \mathbb{R}$. Then

$$e_1 \mapsto (2, 1)$$

$$e_2 \mapsto (3, 1)$$

$$e_3 \mapsto (0, -3)$$

$$f((x, y, z) + \lambda(x', y', z')) = f(x + \lambda x', y + \lambda y', z + \lambda z')$$

$$= (2(x + \lambda x') + 3(y + \lambda y'), (x + \lambda x') + (y + \lambda y') - 3(z + \lambda z'))$$

Let $F = \mathbb{Z}/p\mathbb{Z}$. Write Frobenius: $\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$
 p prime $x \mapsto x^p$

$$(x+y)^p = x^p + \binom{p}{1}x^{p-1}y + \binom{p}{2}x^{p-2}y^2 + \dots + \binom{p}{p-1}xy^{p-1} + y^p$$

and $p \mid \binom{p}{k}$ for $1 \leq k \leq p-1$

$$\Rightarrow (x+y)^p = x^p + y^p$$

$$(\lambda x)^p = \lambda^p x^p$$

$$\text{Frob} : \mathbb{Z}/p\mathbb{Z}[x] \rightarrow \mathbb{Z}/p\mathbb{Z}[x]$$

$$f \mapsto f^p$$

FIT:

$$a^p \equiv a \pmod{p}$$

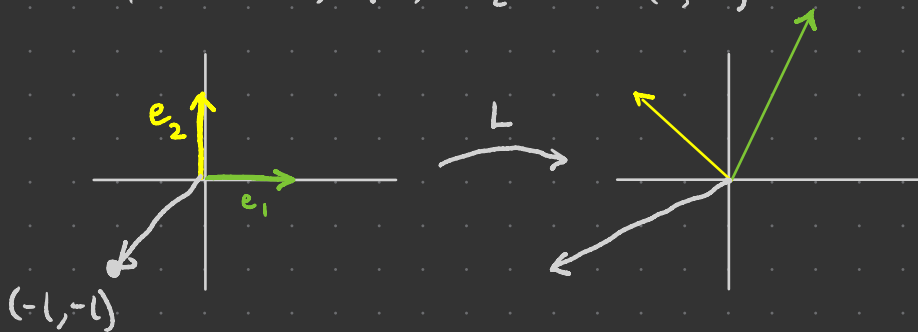
Aha! Frob = id

is a lin trans

Thm (A linear transformation is determined by its action on a basis.)

V, W F-vs's, B a basis of V . For each $b \in B$, suppose $w_b \in W$. Then $\exists!$ linear transformation $L: V \rightarrow W$ s.t. $L(b) = w_b$ for all $b \in B$.

E.g. There is a unique linear trans'n $\mathbb{R}^2 \xrightarrow{L} \mathbb{R}^2$ taking
 $e_1 \mapsto (1, 2)$, $e_2 \mapsto (1, -1)$:



$$\begin{aligned} L(-1, -1) &= \\ L(-e_1 - e_2) &= \\ = -L(e_1) - L(e_2) &= \\ = -(1, 2) - (1, -1) &= (-2, -1) \end{aligned}$$

Pf of Thm For $v \in V$, $\exists! b_1, \dots, b_n \in B$, $\lambda_1, \dots, \lambda_n \in F - \{0\}$

s.t. $v = \sum_{i=1}^n \lambda_i b_i$ Define

$$L(v) = \sum_{i=1}^n \lambda_i w_{b_i}$$

$$\begin{aligned} L(v) &= L(\sum \lambda_i b_i) \\ &= \sum \lambda_i L(b_i) \\ &= \sum \lambda_i w_{b_i} \end{aligned}$$

This is well-defined by uniqueness of b_i, λ_i , and no other expression will work b/c L needs to be linear. $\square$

💡 " L is defined on B and then extended linearly."

In above example $e_1 \mapsto (1, 2)$, $e_2 \mapsto (1, -1)$,

$$\begin{aligned} L(x, y) &= L(xe_1 + ye_2) = xL(e_1) + yL(e_2) \\ &= x(1, 2) + y(1, -1) \\ &= (x+y, 2x-y) \end{aligned}$$

Defn For V, W both F -vs's, let

$$\text{Hom}(V, W) := \{ L: V \rightarrow W \mid L \text{ linear} \}.$$

(Alt notation: $\text{Hom}_F(V, W)$, $\mathcal{L}(V, W)$.)

This is an F -vs via

$$\begin{aligned} f + \lambda g &: V \longrightarrow W \\ v &\longmapsto f(v) + \lambda g(v) \end{aligned}$$

for $f, g \in \text{Hom}(V, W)$, $\lambda \in F$.

If $W = F$, write $V^* := \text{Hom}(V, F)$ and call this the dual of V .
linear functionals