

- Goals
- define bases
 - preservation of linear structure (isomorphism)

Defn A subset $B \subseteq V$ is a basis of V when

- B generates V ($\text{span } B = V$), &
- B is linearly independent

If $B \subseteq V$ is lin ind, then B is a basis for $\text{span } B$

Fact $B \subseteq V$ is a basis iff

- B is a minimal (wrt $\subseteq$) generating set of V
- iff
- B is a maximal (wrt $\subseteq$) lin ind set.

Defn An ordered basis is a basis listed as a sequence:

$b_1, b_2, \dots$

Note If $B = (b_1, \dots, b_n)$ is an ordered basis of V and $v \in V$,
then $\exists!$ $\lambda_1, \dots, \lambda_n \in F$ s.t. $v = \lambda_1 b_1 + \dots + \lambda_n b_n$.
there exists
unique

Defn In the above situation, the coordinates of v wrt B
are $\text{Rep}_B(v) := (\lambda_1, \dots, \lambda_n) \in F^n$

E.g. (1) F^3 has ordered basis $(e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1))$,
its standard ordered basis. Since $(x, y, z) = x e_1 + y e_2 + z e_3$,
the coordinates of (x, y, z) are (x, y, z) . $\therefore$

(2) Take $B' = (e_3, e_2, e_1)$. Then $\text{Rep}_{B'}(x, y, z) = (z, y, x)$
— order matters!

(3) One may check that $B'' = ((1,0,0), (1,1,0), (1,1,1))$ is an ordered basis of F^3 . Since

$$(x,y,z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1),$$

have $\text{Rep}_{B''}(x,y,z) = (x-y, y-z, z)$. For instance,

$$\text{Rep}_{B''}(1,0,3) = (1, -3, 3).$$

(4) Problem Find a basis for $F^{2 \times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in F \right\}$

$$\nabla \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(5) Suppose we know $C = (5, 3), (1, 4)$ is an ordered basis of $\mathbb{R}^2$. Let's determine $\text{Rep}_C(7, -6)$:

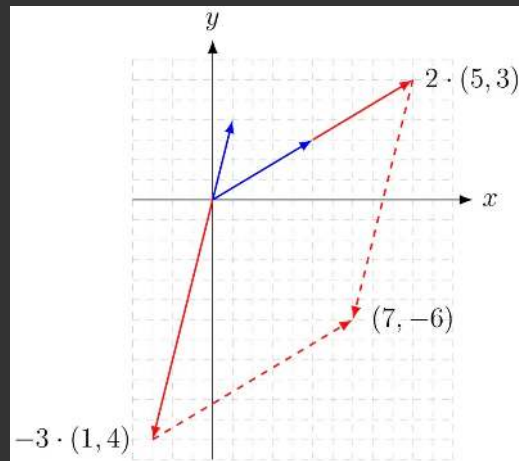
Need $(\lambda, \mu) \in \mathbb{R}^2$ s.t. $\lambda(5, 3) + \mu(1, 4) = (7, -6)$.

$$\begin{aligned} 5\lambda + \mu &= 7 \\ 3\lambda + 4\mu &= -6 \end{aligned} \rightsquigarrow \left(\begin{array}{cc|c} 5 & 1 & 7 \\ 3 & 4 & -6 \end{array} \right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array} \right)$$

so $\text{Rep}_C(7, -6) = (2, -3)$.

$$\begin{aligned} x_1 - x_5 &= 200 \\ x_1 + x_2 &= 50 \end{aligned}$$

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline 1 & 0 & 0 & 0 & -1 & 200 \\ 1 & 1 & 0 & 0 & 0 & 50 \end{array}$$



Given an ordered basis $B = (v_1, \dots, v_n)$ of V , we get inverse bijections

$$V \xrightleftharpoons{\text{Rep}_B} F^n$$

$$\begin{array}{ccc} \lambda_1 v_1 + \dots + \lambda_n v_n = v & \longleftrightarrow & \text{Rep}_B(v) = (\lambda_1, \dots, \lambda_n) \\ \uparrow & & \uparrow \\ x_1 v_1 + \dots + x_n v_n & \longleftrightarrow & (x_1, \dots, x_n) \end{array}$$

But these aren't just bijections! They also preserve linear structure:

$f: V \longrightarrow W$ such that

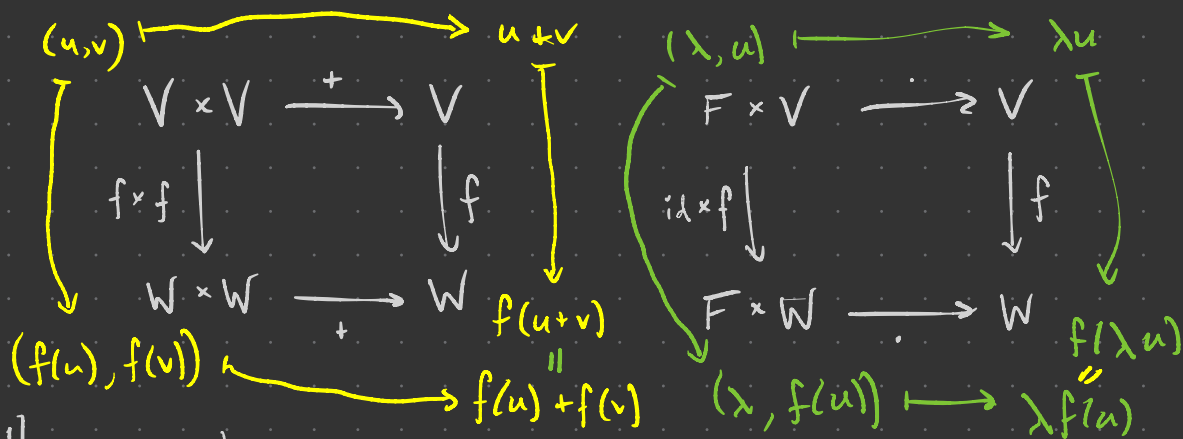
V, W F -vs

- $f(u+v) = f(u) + f(v)$
- $f(\lambda u) = \lambda f(u)$

for all $u, v \in V$, $\lambda \in F$.

Such a function is called a linear transformation.

Diagrammatically:



both commute.

V, W F -vector spaces.

Defn A linear transformation $f: V \longrightarrow W$ is called an isomorphism when it admits a two-sided inverse

$g: W \rightarrow V$ that is also a linear transformation.

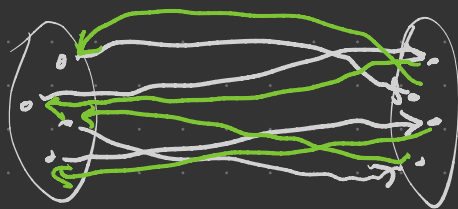
In this case, call V, W isomorphic and write $V \cong W$.

E.g. $\text{Rep}_B: V \xrightarrow{\cong} F^n$ for B any ordered basis of V .
with n elements

Prop A function $f: V \rightarrow W$ is an isomorphism iff $\uparrow$
it is a bijective linear transformation.
 n will work only one
dimension theory

If $(\Rightarrow)$ Assume f is an iso. Then f admits an inverse
lin trans'n. In particular, that's a 2-sided inverse to
 f as a function. By Math 112, f is bijective.

($\Leftarrow$) Assume $f: V \rightarrow W$
is a linear bijection.



Let $g = f^{-1}$ (as a function). Want to show g is linear.

~~Note $g(u + \lambda v) = g(f$~~

Take $u, v \in W, \lambda \in F$. Since f is bij, $\exists u', v' \in V$ s.t.

$f(u') = u, f(v') = v$. Thus $g(u + \lambda v) = g(f(u') + \lambda f(v'))$

$= g(f(u' + \lambda v'))$ since f is linear.

$= u' + \lambda v'$ since $g \circ f = \text{id}_V$

$= g(u) + \lambda g(v)$ (inverses). $\square$