24. IX. 16

Goals · Linear (in)dependence

Future goal: Find "minimal" generating sets (i.e. bases).
To do so, need to know when a generating set is "inefficient".

Fix a field F and F-vs V.

Defn A set $S \subseteq V$ is <u>linearly dependent</u> when $\exists$ distinct $u_1, \dots, u_n \in S$ and $\exists \lambda_1, \dots, \lambda_n \in F$ not all 0 s.t. $\lambda_1 u_1 + \cdots + \lambda_n u_n \overset{\star}{=} 0$.

Call $\star$ a (<u>nontrivial</u>) <u>linear relation</u> among $u_1, \dots, u_n$.
(The trivial linear rel'n is $0u_1 + \cdots + 0u_n = 0$.)

Defn A set $S \subseteq V$ is <u>linearly independent</u> when it is <u>not</u> linearly dependent, i.e.

$u_1, \dots, u_n \in S$ distinct and $\sum \lambda_i u_i = 0 \implies \lambda_1 = \dots = \lambda_n = 0$.

$S \subseteq V$

$0 \in S$ is the vector in V is the add id.

E.g. (0) $\emptyset$ is linearly independent

(1) If $0 \in S$, then S is linearly dependent $\quad 1 \cdot 0 = 0$

(2) Is $S = \{(1,-1,0), (-1,0,2), (-5,3,4)\} \subseteq \mathbb{R}^3$ linearly dependent? Need to find all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ s.t.

$$\lambda_1 (1,-1,0) + \lambda_2 (-1,0,2) + \lambda_3 (-5,3,4) = (0,0,0).$$

I.e.

$$\begin{aligned} \lambda_1 - \lambda_2 - 5\lambda_3 &= 0 \\ -\lambda_1 \quad\quad + 3\lambda_3 &= 0 \\ 2\lambda_2 + 4\lambda_3 &= 0 \end{aligned}$$

Gaussian reduction:

$$\left(\begin{array}{ccc|c} 1 & -1 & -5 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So solutions $\lambda_1 = 3\lambda_3$
$\lambda_2 = -2\lambda_3$

sol'ns: $\{(3\lambda_3, -2\lambda_3, \lambda_3) \mid \lambda_3 \in \mathbb{R}\}$
$= \lambda_3 \cdot (3, -2, 1)$

i.e. $\operatorname{span}_{\mathbb{R}}\{(3,-2,1)\}$, an infinite set,

Have nontriv lin rel'ns such as $(\lambda_3 = 1)$:

$$3(1,-1,-5) - 2(-1,0,3) + (0,2,4) = (0,0,0)$$

so S is linearly dependent.

<u>Prop</u> A subset $S \subseteq V$ is linearly dependent iff $\Leftrightarrow$ $\exists\, v \in S$ s.t. v is a linear combination of vectors in $S \setminus \{v\}$, i.e. $v \in \operatorname{span}(S \setminus \{v\})$.

<u>Pf</u> First note $\emptyset$ is lin ind and $\nexists\, v \in \emptyset$ so can now assume $S \neq \emptyset$.

($\Rightarrow$) Suppose $\sum_{i=1}^{n} \lambda_i u_i = 0$ is a nontrivial linear rel'n among elts of S. WLOG, may assume $\lambda_1 \neq 0$. Then

without loss of generality

$$u_1 = \frac{-\lambda_2}{\lambda_1} u_2 - \frac{\lambda_3}{\lambda_1} u_3 - \cdots - \frac{\lambda_n}{\lambda_1} u_n$$

and we may take $v = u_1$.

Q Does this work for $S = \{0\}$?

$v = 0 \Rightarrow S \setminus \{v\} = \emptyset$

$\Rightarrow 0 \in \operatorname{span} \emptyset$ ✓

($\Leftarrow$) If $v = \lambda_1 u_1 + \cdots + \lambda_n u_n$ with $\lambda_i \in F$, $u_i \in S \setminus \{v\}$, then

distinct

$$0 = \lambda_1 u_1 + \cdots + \lambda_n u_n - v \text{ is a nontrivial linear rel'n}$$

in S.

$\square$

E.g. (3) For all $u \in V \setminus \{0\}$, $\{u\}$ is linearly independent:

$$\lambda u = 0 \text{ for } \lambda \neq 0 \Rightarrow u = \tfrac{1}{\lambda} 0 = 0.$$

(4) Is $S = \{(1,-1,0), (-1,0,2), (0,1,1)\} \subseteq \mathbb{R}^3$ lin ind?

The eq'n $\lambda_1 (1,-1,0) + \lambda_2 (-1,0,2) + \lambda_3 (0,1,1) = (0,0,0)$

is equivalent to

$$\begin{aligned} \lambda_1 - \lambda_2 &= 0 \\ -\lambda_1 + \lambda_3 &= 0 \\ 2\lambda_2 + \lambda_3 &= 0 \end{aligned}$$

and $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

so the unique sol'n is $(\lambda_1, \lambda_2, \lambda_3) = (0,0,0)$.

Thus S is lin ind.

(5) The set $\{1, x, x^2, \ldots\} \subseteq F[x]$ is lin ind.

Problems

(1) Is $\{\sin, \cos\} \subseteq \mathbb{R}^{\mathbb{R}}$ lin ind?

(2) Prove that for $S \subseteq T \subseteq V$, T lin ind $\Rightarrow S$ lin ind.

(3) Is $\{1, \sin^2, \cos^2\} \subseteq \mathbb{R}^{\mathbb{R}}$ lin ind? Dep.

(1) $a \sin x + b \cos x = 0$

$x=0$: $a \cancel{\sin} 0 + b \cancel{\cos 0}^{\,1} = 0$

$b = 0$

$x = \frac{\pi}{2} \Rightarrow a = 0$

(2) Contrapositive $A \Rightarrow B$ equiv $\neg B \Rightarrow \neg A$