

Goals

- finite dimension is well-defined
- learn to compute $\dim V$

Defn A vector space is finite dimensional when it has a basis with finitely many elements.

E.g.

- $F^n, F^{m \times n}$ have bases of cardinality $n, mn < \infty$, resp.
- $F[x], \mathbb{R}^{\mathbb{R}}$ are infinite dimensional

Defn If V is a finite dimensional F -vs, then the dimension of V , denoted $\dim V = \dim_F V$, is the cardinality of any basis of V .



Is this well-defined?

Exchange lemma Suppose $B = \{v_1, \dots, v_n\}$ is a basis of V and

$w = \sum_{i=1}^n \lambda_i v_i$ with $\lambda_i \in F$ not all 0. If $\lambda_l \neq 0$ for some $l \in \{1, \dots, n\}$, then $B' = (\underbrace{B - \{v_l\}}_{v_l, w \text{ exchanged}}) \cup \{w\}$ is also a basis of V .

Thus $B' = \{w, v_2, v_3, \dots, v_n\}$

Pf First show B' lin ind. WLOG, $l=1$. Suppose $\mu w + \mu_2 v_2 + \dots + \mu_n v_n = 0$.

Subbing $\star$, $\mu \left(\sum_{i=1}^n \lambda_i v_i \right) + \mu_2 v_2 + \dots + \mu_n v_n = 0$

$$\Leftrightarrow \mu \lambda_1 v_1 + (\mu \lambda_2 + \mu_2) v_2 + \dots + (\mu \lambda_n + \mu_n) v_n = 0.$$

Since B lin ind, $\cancel{\mu \lambda_1}^0 = \cancel{\mu \lambda_2 + \mu_2}^0 = \dots = \cancel{\mu \lambda_n + \mu_n}^0 = 0$.

Since $\lambda_1 \neq 0$, know $\mu = 0$, whence $\mu_2 = \dots = \mu_n = 0$.

Thus $B' = \{w, v_2, \dots, v_n\}$ is lin ind.

$$w = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$$

(still with $\lambda_1 \neq 0$)

Now show $\text{span } B' = V$. Solving for v_1 in $\star$ gives

$$v_1 = \frac{1}{\lambda_1} w - \frac{\lambda_2}{\lambda_1} v_2 - \dots - \frac{\lambda_n}{\lambda_1} v_n$$

For $v \in V$, B a basis $\Rightarrow$

$$v = \mu_1 v_1 + \dots + \mu_n v_n \text{ for some } \mu_i \in F.$$

Subbing in $\star$ gives

$$v = \mu_1 \left(\frac{1}{\lambda_1} w - \frac{\lambda_2}{\lambda_1} v_2 - \dots - \frac{\lambda_n}{\lambda_1} v_n \right) + \mu_2 v_2 + \dots + \mu_n v_n$$

$$= \frac{\mu_1}{\lambda_1} w + \left(\mu_2 - \frac{\mu_1 \lambda_2}{\lambda_1} \right) v_2 + \dots + \left(\mu_n - \frac{\mu_1 \lambda_n}{\lambda_1} \right) v_n$$

$$\in \text{span } B'$$

Thus $\text{span } B' = V$ and we've already seen B' lin ind, so B' is a basis. $\square$

Thm

In a finite dimensional vector space V , every basis has the same cardinality.

Pf Among all bases of V , let $B = \{v_1, \dots, v_n\}$ be one of minimal cardinality. Let $C = \{w_1, w_2, \dots\}$ be another basis of V .

WTS: $|C| = |B|$. Know $n = |B| \leq |C|$.

Idea: Use the exchange lemma to swap n elts of C into B , while maintaining basis status.

let $B_0 = B$, take $w_1 \in C$. By the exchange lemma,
get new basis B_1 by swapping w_1 in for some $v_l \in B_0$.
WLOG, $l=1$ and $B_1 = \{w_1, v_2, \dots, v_n\}$ is a basis of V .
Take $w_2 \in C \setminus \{w_1\}$.
Since B_1 is a basis,

$$w_2 = \lambda_1 w_1 + \lambda_2 v_2 + \dots + \lambda_n v_n \text{ for some } \lambda_i \in F.$$

Since w_1, w_2 are lin ind, some $\lambda_l, l \geq 2$ is nonzero.

WLOG, $l=2$ and we can exchange to get $B_2 = \{w_1, w_2, v_3, \dots, v_n\}$
a basis of V .

Continuing in this fashion, eventually get

$B_n = \{w_1, \dots, w_n\}$ is a basis of V

in

C

In fact, $B_n = C$ ^{because otherwise} b/c o/w $w_{n+1} \in \text{span } B_n$

$\Rightarrow C$ lin dep. $\square$

Cor dimension of fin dim vs's is well-defined

Cor If V is a fin dim vs, $S \subseteq V$ lin ind, then we may extend S to a basis of V by adding some $\dim V - |S|$ elements.

Pf Apply the "basis production algorithm" from the theorem's proof. $\square$

Cor If V fin dim vs and $T \subseteq V$ generates V , then some subset $S \subseteq T$ is a basis of V . $\square$

$\hookrightarrow$ Suppose $S \subseteq V$ lin ind but $\text{span } S \subsetneq V$. Then
 take $w \in V \setminus \text{span } S$. Claim $S \cup \{w\}$ is lin ind.

This is the case iff $\forall v \in S \cup \{w\}$, v is not
 a lin combo of elts of $(S \cup \{w\}) \setminus \{v\}$.

True since S lin ind + $w \in V \setminus \text{span } S$. $\square$

#6 $S = \{(a,b), (c,d)\}$ lin ind? $\sim$ " $ad-bc$ " = $1 \cdot 1 - 0 \cdot 0$
 $= 1$

$\Leftrightarrow \begin{pmatrix} a & c & | & 0 \\ b & d & | & 0 \end{pmatrix} \xrightarrow{\text{row ops}} \begin{pmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{pmatrix}$

- row ops
- what happens to $ad-bc$ when you apply a row op?