

- Goals
- Two operations & eight properties of vector spaces
 - Become familiar with standard and nonstandard examples of vector spaces
 - Define subspaces

Fix a field F .

Defn A vector space over F is a set V together with operations

$$+ : V \times V \longrightarrow V \quad \text{vector addition}$$

$$\cdot : F \times V \longrightarrow V \quad \text{scalar multiplication}$$

such that for all $u, v, w \in V$, $\lambda, \mu \in F$

(1) $u + v = v + u$ commutativity of $+$

(2) $u + (v + w) = (u + v) + w$ associativity of $+$

(3) $\exists 0 \in V$ s.t. $0 + u = u$ additive identity

(4) $\exists -u \in V$ s.t. $u + (-u) = 0$ additive inverse

(5) $\overset{F}{\rightarrow} 1 \cdot u = u$ scalar multiplicative identity

(6) $\lambda(\mu v) = (\underbrace{\lambda\mu}_{\text{product in } F}) \cdot v$ associativity of scalar mult'n

(7) $\lambda \cdot (u + v) = (\lambda u) + (\lambda v)$ distribution of scalar mult'n over vector add'n

(8) $(\lambda + \mu) \cdot u = (\lambda u) + (\mu u)$ distribution of scalar mult'n over field add'n

Note (1) - (4) make $(V, +)$ an Abelian group



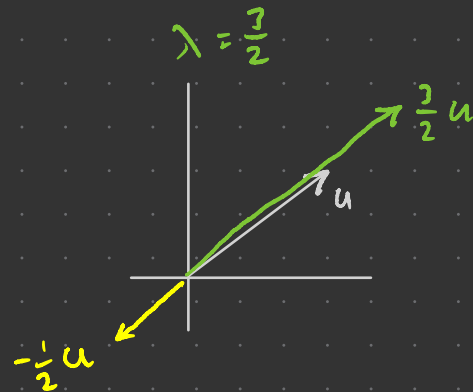
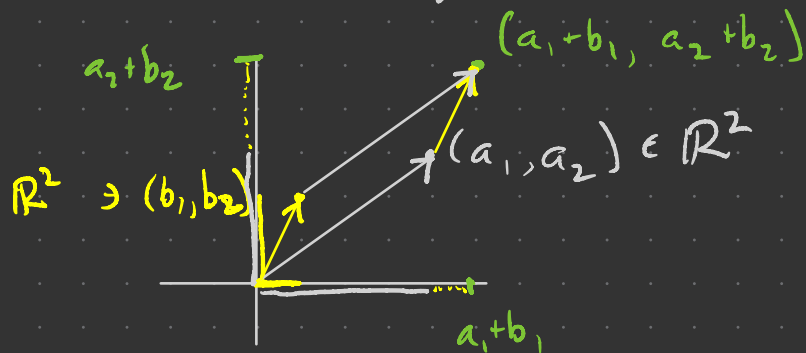
cannot, in gen'l, multiply vectors!

E.g. • For $n \in \mathbb{N}$, $F^n = \underbrace{F \times \dots \times F}_{n \text{ times}} = \{(a_1, a_2, \dots, a_n) \mid a_i \in F, 1 \leq i \leq n\}$

with $(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$

and $\lambda \cdot (a_1, \dots, a_n) = (\lambda a_1, \dots, \lambda a_n)$

• For $n=2$, $F=\mathbb{R}$, get Euclidean plane:



• $F = \mathbb{Z}/3\mathbb{Z}$, $n=4$: $(1, 0, 2, 1) + 2(0, 0, 1, 1) = (1, 0, 2, 1) + (0, 0, 2, 2)$
 $= (1, 0, 1, 0) \in F^4$

• $\mathbb{C}$ is an $\mathbb{R}$ -vs via $\lambda(a+bi) = (\lambda a) + (\lambda b)i$, $a, b, \lambda \in \mathbb{R}$.

• If $F \subseteq L$ are both fields, then L is an F -vs, e.g.

$\mathbb{R}$ is a $\mathbb{Q}$ -vs.

• $F^{m \times n} := \{ m \times n \text{ matrices with entries in } F \}$

$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \mid a_{ij} \in F, 1 \leq i \leq m, 1 \leq j \leq n \right\}$$

For $A \in F^{m \times n}$, write A_{ij} for its ij -th entry. Then for $A, B \in F^{m \times n}$, $\lambda \in F$

$$(A+B)_{ij} := A_{ij} + B_{ij}$$

$$(\lambda A)_{ij} := \lambda A_{ij}$$

So for $F = \mathbb{Q}$, $m=2$, $n=3$,

$$2 \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 3 \end{pmatrix} - 5 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 8 & 1 \end{pmatrix}$$

• If S is any set, then $F^S := \{f \mid f: S \rightarrow F \text{ a function}\}$

is an F -vs where for $f, g \in F^S$, $\lambda \in F$:

$$\begin{array}{ccc} f+g: S & \longrightarrow & F \\ s & \longmapsto & f(s) + g(s) \end{array}$$

$$\lambda \cdot f: S \longrightarrow F$$

$$s \longmapsto \lambda f(s)$$

Note For $n \in \mathbb{N}$, let $\underline{n} := \{1, 2, \dots, n\}$. Then

$$F^{\underline{n}} \overset{\text{isomorphic}}{=} F^n$$

$$\begin{array}{ccc} f & \longmapsto & (f(1), f(2), \dots, f(n)) \\ \downarrow \text{ } i & & \\ \downarrow \text{ } i & \longleftarrow & (a_1, \dots, a_n) \\ F & a_i & \end{array}$$

$$F^{\underline{m} \times \underline{n}} = F^{m \times n}$$

$$\underline{m} \times \underline{n} = \{(i, j) \mid i \in \underline{m}, j \in \underline{n}\}$$

$$\begin{array}{ccc} f & \longmapsto & (f(i, j))_{i, j} \\ (i, j) & & \\ \downarrow \text{ } i, j & \longleftarrow & A \\ A_{i, j} & & \end{array}$$

Note $\mathbb{R}^{\mathbb{R}} = \{f \mid f: \mathbb{R} \rightarrow \mathbb{R}\}$ is a vector space that calculus students love.

Q Other examples?



Subspaces

$$\mathbb{R}^{\mathbb{R}}$$

$\cup$

$$C(\mathbb{R}) = \{f \mid f: \mathbb{R} \rightarrow \mathbb{R} \text{ is continuous}\}$$

$\cup$

$$C'(\mathbb{R}) = \{f \mid f: \mathbb{R} \rightarrow \mathbb{R} \text{ with } f' \text{ continuous}\}$$

$\cup$

$$C^{\infty}(\mathbb{R}) = \{f \mid f: \mathbb{R} \rightarrow \mathbb{R} \text{ has continuous derivatives of all orders}\}$$

$\cup$

$$\mathbb{R}[x] = \{f \mid f: \mathbb{R} \rightarrow \mathbb{R} \text{ is a polynomial function}\}$$

Each of these satisfies $f, g \in W \Rightarrow f+g \in W, \lambda f \in W, 0 \in W$

In fact, each is a vector space in its own right.