

- Goals
- Define echelon & reduced echelon forms
 - Every system has a unique reduced echelon form (REF) (w/ algorithm)
 - Solution sets from REF.

Defn A matrix is in echelon form when ^{leave out augmentation}

(1) all rows of just 0's are at the bottom

(2) the pivot of each nonzero row is strictly to the right of the pivot of the previous row.
^{first nonzero term in the row}

E.g.

$$\begin{pmatrix} 1 & * & * & * & * \\ 0 & 0 & 2 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Prop We can put any matrix in echelon form via row ops of type (2) & (3).

swap — add λr_j to r_i , $\lambda \in F$

Pf (by algorithm)

(1) Swap rows to get (any) leftmost pivot in top row

(2) Use (3) to get 0's below this pivot

(3) Repeat previous steps with the matrix obtained by deleting first row & column. $\square$

E.g. System:

$$\begin{array}{rcl} & 2x_3 + 6x_4 & = 0 \\ x_1 + 2x_2 + x_3 + 3x_4 & = 1 \\ 2x_1 + 4x_2 + 3x_3 + 9x_4 + x_5 & = 5 \end{array}$$

Augmented matrix:

$$\left(\begin{array}{ccccc|c} 0 & 0 & 2 & 6 & 0 & 0 \\ 1 & 2 & 1 & 3 & 0 & 1 \\ 2 & 4 & 3 & 9 & 1 & 5 \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 2 & 4 & 3 & 9 & 1 & 5 \end{array} \right)$$

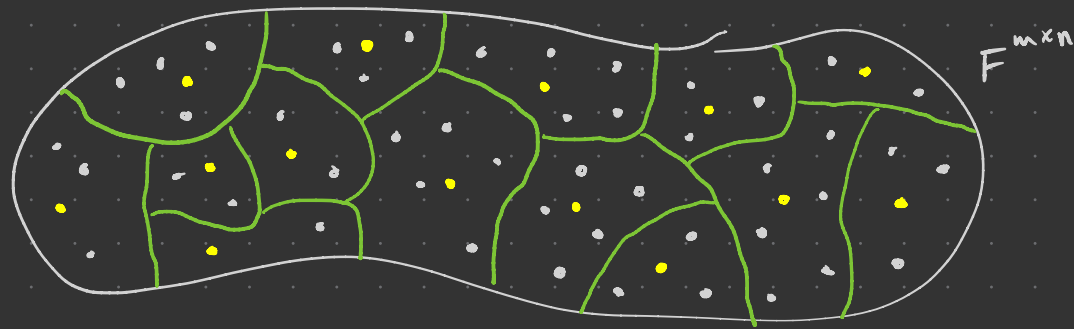
$$\begin{aligned} r_3 \rightarrow r_3 - 2r_1 & \rightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 3 \end{array} \right) & r_3 \rightarrow r_3 - \frac{1}{2}r_2 & \rightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array} \right) \end{aligned}$$

REF

Now in echelon form!

Defn A matrix is in reduced echelon form when it is in echelon form, each pivot is 1, and above and below each pivot the column is all 0's.

Thm Every matrix is equivalent (via row operations) to a unique matrix in REF.



- not in REF
 - in REF
- equivalence classes

Pf To produce a REF representative, first pass to echelon form, then scale rows by $\frac{1}{p_i}$ where p_i = pivot of i -th row, then use

(3) to clear the column above and below each pivot.

Uniqueness: One. III. 2.6. $\square$

E.g. $\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array} \right) \xrightarrow{r_2 \rightarrow \frac{1}{2}r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array} \right)$

$\xrightarrow{r_1 \rightarrow r_1 - r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array} \right)$

pivots $\longleftrightarrow$ pivot variables x_1, x_3, x_5

Other variables are free

Solution set $\left\{ (x_1, \dots, x_5) \in F^5 \mid \begin{cases} x_1 + 2x_2 = 1 \\ x_3 + 3x_4 = 0 \\ x_5 = 3 \end{cases} \right\}$

Solving for pivots:

$$\begin{aligned} x_1 &= 1 - 2x_2 \\ x_3 &= -3x_4 \\ x_5 &= 3 \end{aligned}$$

This gives the parametric form of the solution set

$$\left\{ (\underline{1 - 2x_2}, \underline{x_2}, \underline{-3x_4}, \underline{x_4}, \underline{3}) \mid x_2, x_4 \in F \right\}$$

and vector form

pivots
(now in terms of free)

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} \mid x_2, x_4 \in F \right\}$$

Problem Find all parabolas $f(x) = ax^2 + bx + c$ passing through $(1, 4)$ and $(3, 6)$.

Solution We know $4 = a + b + c$ resulting in the
 $6 = 9a + 3b + c$

augmented matrix $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 9 & 3 & 1 & 6 \end{array} \right)$. We now apply

Gaussian reduction:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 9 & 3 & 1 & 6 \end{array} \right) \xrightarrow{r_2 \rightarrow r_2 - 9r_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -6 & -8 & -30 \end{array} \right)$$

$$r_2 \rightarrow -\frac{1}{6}r_2 \xrightarrow{\quad} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 5 \end{array} \right) \xrightarrow{r_1 \rightarrow r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & -1 \\ 0 & 1 & \frac{4}{3} & 5 \end{array} \right)$$

$$\left\{ \left(\frac{1}{3}c - 1, -\frac{4}{3}c + 5, c \right) \mid c \in \mathbb{R} \right\}$$