

Goals

- Algebra & geometry of systems of linear eq's
- Intro to Gaussian elimination / row reduction

Note Today, $F = \mathbb{R}$. Non-geometry work over any field.

Example 1 Solve $\begin{cases} 3x + 2y = 5 \\ 2x - y = 1 \end{cases}$ } system of linear eq's
in 2 variables x, y

Multiply 2nd eq'n by 2, then add eq's to eliminate y :

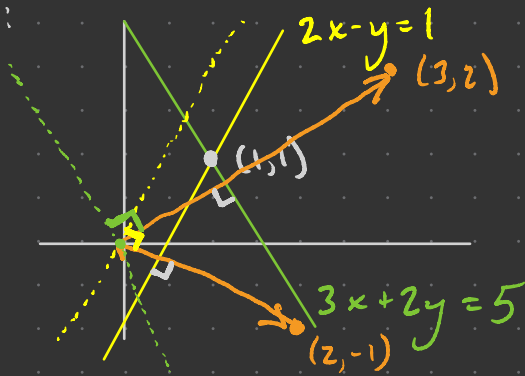
$$\begin{array}{r} 3x + 2y = 5 \\ + \quad 4x - 2y = 2 \\ \hline 7x = 7 \Rightarrow x = 1 \end{array}$$

Sub $x=1$ into first eq'n to get

$$3 \cdot 1 + 2y = 5 \Rightarrow y = 1.$$

Thus $x=y=1$ is the unique solution.

Geometrically:

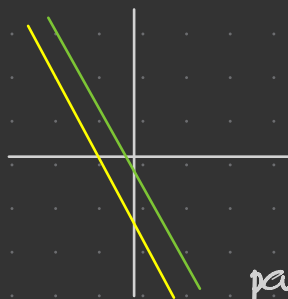


Question What is the relationship b/w $2x-y=1$ and $(2, -1)$?
 $3x+2y=5$ and $(3, 2)$?

Example 2 System $-9x - 3y = 6$ (1)
 $3x + y = -2$ (2)

Here (1) = $-3 \cdot$ (2) so they have the same solutions.

Example 3 System $-9x - 3y = 6 \xrightarrow{\cdot \frac{-1}{3}} 3x + y = -2$
 $3x + y = -1$



Cannot both be true!
Thus no solutions.

parallel lines — same normal vectors!

Gaussian elimination

Idea: Transform a system into a new one with

(a) same set of solutions

(b) evident solutions

Legal transformations:

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(1) Multiply an eq'n by $\lambda \in F^\times = \{x \in F \mid x \neq 0\}$

(2) Swap two eq'ns

(3) Add a multiple of one eq'n to another

Discuss Why are these "legal"?

Augmented matrices

$$x + 2y + z = 0$$

$$x \quad \quad + z = 4$$

$$x + y + 2z = 1$$

$$\leadsto \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 2 & 1 \end{array} \right)$$

Just record
the coefficients!

Notation: $r_i = i$ -th row of augmented matrix

Now eliminate:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 2 & 1 \end{array} \right)$$

$$r_2 \rightarrow r_2 - r_1$$

$$r_3 \rightarrow r_3 - r_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 0 & 4 \\ 0 & -1 & 1 & 1 \end{array} \right)$$

elim x from 2nd eqn

elim x from 3rd eqn

$$r_2 \rightarrow -\frac{1}{2}r_2 \rightarrow \begin{pmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & 0 & | & -2 \\ 0 & -1 & 1 & | & 1 \end{pmatrix} \xrightarrow{r_3 \rightarrow r_3 + r_2} \begin{pmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & -1 \end{pmatrix}$$

(solve for y in 2nd eqn
elim y from 3rd eqn

$$r_1 \rightarrow -2r_2 + r_1 \rightarrow \begin{pmatrix} 1 & 0 & 1 & | & 4 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & -1 \end{pmatrix} \xrightarrow{r_1 \rightarrow -r_3 + r_1} \begin{pmatrix} 1 & 0 & 0 & | & 5 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & -1 \end{pmatrix}$$

elim y from 1st eqn

Thus the unique sol'n is $(x, y, z) = (5, -2, -1)$.

Question Check this!

Example 4 Solve $x + 2y + z = 0$

$$x \quad \quad + z = 4$$

$$x + y + z = 1$$

Then

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{array} \right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -1 \end{array} \right)$$

and thus $x=4$, $y=-2$, $0=-1$ ☹

There are no sol'ns to this system.

Example 5 Another small modification:

$$x + 2y + z = 0$$

$$x + z = 4$$

$$x + y + z = 2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 2 \end{array} \right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

So $x + z = 4$

$$y = -2$$

$$0 = 0$$

$$\Rightarrow \text{sol'n set } \{(x, -2, 4-x) \mid x \in \mathbb{R}\}$$

a line in $\mathbb{R}^3$.

Note Each "row reduction" ended in "reduced echelon form".

To do: (a) define this

(b) prove Gaussian reduction always gives this form

(c) understand sol'n sets from REF.