Math 111: Calculus Practice Exam 2 — Solutions

These are written out in full. On the exam you would not need this much prose — but you would need the *reasons*, and every reason here is one that earns marks.

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1. Structure first

For each function, first say in a few words how it is built (a sum? a product? a quotient? a composition, and of what?), then differentiate it. You need not simplify.

(a) $f(x) = x^2 \cos x$

f is a **product** of x^2 and $\cos x$. Product rule: $f'(x) = 2x\cos x - x^2 \sin x$.

(b) $g(x) = \dfrac{e^x}{x+1}$

g is a **quotient**, e^x over $x+1$. Quotient rule:

$$g'(x) = \frac{e^x(x+1) - e^x \cdot 1}{(x+1)^2} = \frac{x\,e^x}{(x+1)^2}.$$

(c) $h(x) = \sqrt{1+\sin(3x)}$

h is a **composition**, twice over: the square root is outside, $1+\sin(3x)$ is inside, and inside *that*, $3x$ sits inside $\sin$. Chain rule, applied from the outside in:

$$h'(x) = \frac{1}{2\sqrt{1+\sin(3x)}} \cdot \cos(3x) \cdot 3 = \frac{3\cos(3x)}{2\sqrt{1+\sin(3x)}}.$$

(d) $k(x) = \ln(1+x^2)$

k is a **composition**: $\ln$ outside, $1+x^2$ inside. Chain rule: $k'(x) = \dfrac{1}{1+x^2} \cdot 2x = \dfrac{2x}{1+x^2}$.

(e) *Find the tangent line to the graph of f at $x = \pi$, and use it to estimate $f(3)$. (You may leave π in your answer.)*

$f(\pi) = \pi^2 \cos\pi = -\pi^2$ and $f'(\pi) = 2\pi\cos\pi - \pi^2 \sin\pi = -2\pi$. So the tangent line is

$$L(x) = -\pi^2 - 2\pi\,(x - \pi),$$

and

$$f(3) \approx L(3) = -\pi^2 - 2\pi(3-\pi) = \pi^2 - 6\pi \approx -8.98.$$

(For comparison, $f(3) = 9\cos 3 \approx -8.91$.)

2. A balloon, a barometer, and two tables

A weather balloon rises. Its altitude $a(t)$, in meters, is recorded every five minutes:

t (minutes)	0	5	10	15	20
$a(t)$ (m)	0	400	750	1050	1300

Air pressure $P(a)$, in kilopascals, depends only on altitude a:

a (m)	0	500	1000	1500	2000
$P(a)$ (kPa)	101.3	95.5	90.0	84.8	79.9

The pressure the balloon experiences at time t is $H(t) = P\big(a(t)\big)$. There are no formulas; the tables are all you have.

It pays to write out the average rates first. For the altitude table, over each five-minute stretch: $80, 70, 60, 50$ m/min — the climb is slowing. For the pressure table, over each 500-m stretch: -0.0116, -0.0110, -0.0104, -0.0098 kPa/m — pressure falls more slowly the higher you go.

(a) *Estimate $a'(10)$, with units. Then give one estimate of $a'(10)$ that is certainly too large and one that is certainly too small, and say what you are assuming about a' in order to be sure.*

Central difference: $a'(10) \approx \dfrac{a(15) - a(5)}{15 - 5} = \dfrac{1050 - 400}{10} = 65$ m/min.

The averages $80, 70, 60, 50$ are decreasing, so a' appears to be decreasing. If it is, then every instantaneous rate on $[5, 10]$ is at least $a'(10)$, so the backward estimate $\frac{750-400}{5} = 70$ m/min is **too large**; and every rate on $[10, 15]$ is at most $a'(10)$, so the forward estimate $\frac{1050-750}{5} = 60$ m/min is **too small**. The assumption doing the work is that a' is decreasing throughout $[5, 15]$, not just at the sampled instants.

(b) *Estimate $P'(750)$, with units. Say in one sentence what the sign of this number means physically.*

$a(10) = 750$ lies between the table entries 500 and 1000, so the natural estimate is the secant across them:

$$P'(750) \approx \frac{P(1000) - P(500)}{1000 - 500} = \frac{90.0 - 95.5}{500} = -0.011 \text{ kPa per meter.}$$

The sign is negative because pressure *drops* as the balloon rises: each extra meter of altitude costs about 0.011 kPa.

(c) *Estimate $H'(10)$, with units. Explain why the two numbers you combine should be combined the way you combined them — in terms of what happens to the balloon over the next minute, not by naming a rule.*

$H'(10) \approx P'(750) \cdot a'(10) \approx (-0.011)(65) \approx -0.72$ kPa per minute.

Why multiply: over the next minute the balloon climbs about 65 meters (that is what $a'(10) = 65$ m/min says). Each of those meters lowers the pressure by about 0.011 kPa (that is what $P'(750)$ says). So the pressure the balloon feels drops by about $65 \times 0.011 \approx 0.72$ kPa in that minute. The units agree: (kPa/m)×(m/min) = kPa/min, and the meters cancel exactly because the first rate is *per meter of altitude* and the second *produces* meters of altitude.

(d) *A classmate computes $H'(10)$ as $P'(10) \cdot a'(10)$. Say precisely what is wrong, and why the number $P'(10)$ is not even the kind of quantity that belongs here.*

$P'(10)$ is the rate of change of pressure at an *altitude* of 10 meters, because P is a function of altitude, not time. The balloon is not at 10 m at time 10; it is at 750 m. The derivative of the outside function has to be evaluated at the inside function's value, $a(10) = 750$, not at the time 10. Beyond the wrong number, $P'(10)$ answers a question nobody asked — what happens to pressure near the ground — and so cannot be the right factor regardless of its value.

3. A melting snowball

A snowball is melting. Assume it stays a perfect sphere, and that its volume is decreasing at a steady 2 cubic centimeters per minute. Recall $V = \frac{4}{3}\pi r^3$ and surface area $S = 4\pi r^2$.

(a) *Draw and label a picture, and write down, with units, every rate you know and the rate you want.*

A sphere of radius r (cm), with r changing in time. Known: $\dfrac{dV}{dt} = -2$ cm^3/min (negative because the volume is decreasing). Wanted: $\dfrac{dr}{dt}$ when $r = 3$ cm, and then $\dfrac{dS}{dt}$ at the same moment.

(b) *How fast is the radius shrinking at the moment the radius is 3 cm?*

From $V = \frac{4}{3}\pi r^3$, differentiating with respect to time (chain rule, since r depends on t):

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \Longrightarrow \qquad \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{-2}{4\pi r^2}.$$

At $r = 3$: $\dfrac{dr}{dt} = \dfrac{-2}{36\pi} = -\dfrac{1}{18\pi} \approx -0.018$ cm/min. The radius is shrinking by about two hundredths of a centimeter each minute.

(c) *How fast is the surface area shrinking at that moment?*

From $S = 4\pi r^2$: $\dfrac{dS}{dt} = 8\pi r \dfrac{dr}{dt}$. At $r = 3$, using (b):

$$\frac{dS}{dt} = 8\pi \cdot 3 \cdot \Big(-\frac{1}{18\pi}\Big) = -\frac{24}{18} = -\frac{4}{3} \text{ cm}^2\text{/min}.$$

(Equivalently, $\frac{dS}{dt} = \frac{2}{r}\frac{dV}{dt}$, which is a tidy general fact.)

(d) *Is the radius shrinking faster or slower when the snowball is smaller? Answer from your formula in (b), then say in a sentence why that makes physical sense.*

From (b), $\dfrac{dr}{dt} = -\dfrac{2}{4\pi r^2}$: the magnitude grows as r shrinks. So the radius shrinks **faster** when the snowball is smaller. Physically, the same 2 cm^3 of ice is lost each minute, and on a small sphere that volume is a thicker shell than on a big one — the surface area $4\pi r^2$ over which the loss is spread is smaller, so each minute's loss bites deeper into the radius.

4. All about f'

A function f is differentiable everywhere, and its derivative is $f'(x) = (x-2)(x+1)^2$. You are not told f itself.

(a) *Find all critical numbers of f.*

f' is a polynomial, so it is defined everywhere; the critical numbers are where $f'(x) = 0$: $\boxed{x = -1}$ and $\boxed{x = 2}$.

(b) *Make a sign chart for f', and use the first derivative test to classify each critical number as a local maximum, a local minimum, or neither. Justify each from the chart.*

The factor $(x+1)^2$ is never negative, so the sign of f' is the sign of $x - 2$:

	$x < -1$	$-1 < x < 2$	$x > 2$
$x - 2$	$-$	$-$	$+$
$(x+1)^2$	$+$	$+$	$+$
f'	$-$	$-$	$+$

At $x = 2$, f' changes from negative to positive: f falls, then rises, so f has a **local minimum** at 2. At $x = -1$, f' is negative on both sides: no sign change, so **neither** $-$ f is decreasing through $x = -1$.

(c) *Compute f'' and apply the second derivative test at each critical number. Where the test is silent, say so, and say which test settles the matter.*

Product rule: $f''(x) = (x+1)^2 + (x-2)\cdot 2(x+1) = (x+1)\big[(x+1)+2(x-2)\big] = (x+1)(3x-3) = 3(x+1)(x-1)$.

At $x = 2$: $f''(2) = 3 \cdot 3 \cdot 1 = 9 > 0$, so f is concave up there and the second derivative test confirms the **local minimum**. At $x = -1$: $f''(-1) = 0$, so the test is **silent** $-$ it gives no information either way. The first derivative test in (b) settles it: no extremum.

(d) *A classmate writes: "$f'(-1) = 0$, and f' is negative on both sides of -1, so the graph of f turns around at $x = -1$ and f has a local maximum there." Find the first wrong step, say why it is wrong, and describe what the graph of f actually does near $x = -1$.*

The first wrong step is "**so the graph of f turns around at $x = -1$.**" A function turns around where its derivative *changes sign*; a derivative that is negative on both sides of a point describes a function that is falling on both sides, which is the opposite of turning around. What actually happens: f decreases as it approaches $x = -1$, flattens out for an instant (slope exactly 0), and keeps decreasing. Think of $-(x+1)^3$ near -1. There is no local maximum, and no local minimum either.

(e) *On axes of your own, sketch a possible graph of f on $[-3, 4]$, consistent with everything above. Mark the critical points.*

A sketch consistent with all of the above: decreasing on $(-\infty, 2)$ with a flat step at $x = -1$ (concave down to the left of -1 and concave up between -1 and 1, since f'' changes sign at ± 1), a minimum at $x = 2$, increasing after. The vertical position is free, since f is only known up to a constant.

