Math 111: Calculus Homework 3

Due Friday, October 30 by 10:00p.m. via Gradescope

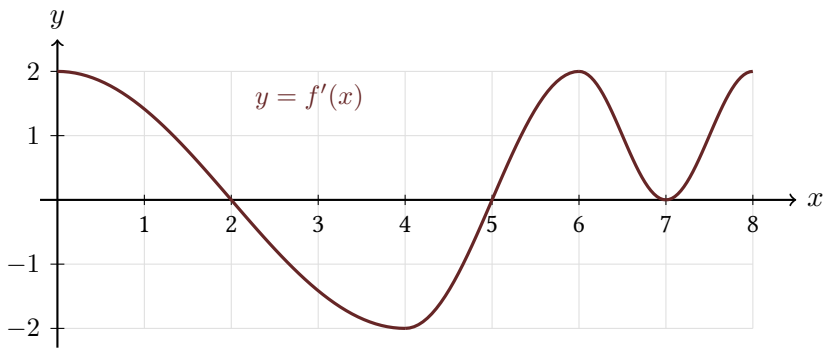
Covers Active Calculus §3.3, §3.4, and §3.6. Problems 1 and 4 use only §3.3, which we cover on Monday, October 12, so you can do them before the break. Problem 2 uses §3.4 (Monday, October 26). Problem 3 can be done with the tools of §3.3 if you follow the steps given; §3.6 (Wednesday, October 28) will make it feel routine. **No work is expected over fall break.**

Before you start. Write in sentences and paragraphs, as an explanation another student in this course could read and follow. Draw diagrams large and label them. You are encouraged to work with classmates; cite everyone you worked with, and write your solutions independently — duplicated solutions are an Honor Principle violation. Each problem is marked holistically on the 0–5 scale, with up to two further points for the quality of the writing. If you need an extension, use the extension form *before* the deadline.

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1. Everything about f from a picture of f'

Below is the graph of the *derivative* f' of a function f on $[0, 8]$. This is not a graph of f. No formula is given for either.



(a) Find every critical number of f in $(0, 8)$. Classify each as a local maximum, a local minimum, or neither, and justify each classification from the graph of f'.

(b) On which intervals is f concave up, and on which is it concave down? Where does f have inflection points? Explain how you read each answer off the graph of f'.

(c) One of the critical numbers you found in (a) is also an inflection point. Describe in words what the graph of f looks like near that point, and name a simple formula whose graph behaves the same way near 0.

(d) Where on $[0, 8]$ does f take its *largest* value? Your work in (a) should narrow it down to two candidates but not decide between them. Say what the two candidates are, say exactly what further information about f would decide it, and then make your best guess from the picture of f', explaining what in the picture you are using. (We will have a precise tool for this in November.)

(e) Suppose $f(0) = 1$. Sketch a graph of f on $[0, 8]$ that is consistent with everything above. Mark the critical points and inflection points.

2. A family of functions

For each constant a, let

$$g_a(x) = x + \frac{a}{x}, \qquad x > 0.$$

Keep a as a letter throughout; do not pick a value unless a part asks you to.

(a) Suppose $a > 0$. Find the critical number(s) of g_a on $(0, \infty)$ in terms of a, and classify each. Then compute g_a'' and say what it tells you about the shape of the graph.

(b) Still with $a > 0$: as a varies, the lowest point on the graph of g_a moves. Show that all of these lowest points lie on a single line, and find its equation. On one set of axes, sketch g_1, g_4, and g_9 together with that line.

(c) Now suppose $a < 0$. Does g_a have any critical numbers on $(0, \infty)$? Describe the shape of the graph of g_a in words (increasing or decreasing? concave up or down?), justifying each claim from g_a' and g_a''. What happens when $a = 0$?

(d) Find the value of a for which the lowest point of g_a occurs at $x = 3$, and say what that lowest value is.

3. Why aren't cans shaped like that?

A cylindrical can with radius r and height h holds volume $V = \pi r^2 h$. Its metal consists of a top and a bottom (each a disk of area πr^2) and a side (area $2\pi r h$). Throughout, the volume V is a fixed positive constant.

(a) Use $V = \pi r^2 h$ to write the total surface area S as a function of r alone. Find the value of r that makes S as small as possible, explain how you know it is a minimum and not a maximum, and show that at this radius the height equals the diameter.

(b) In 1987 a mathematics professor wrote to the Carnation company asking why their cans are not shaped that way. The company's reply is reproduced in Peter Keep's *Discover Calculus*: discovercalculus.com/web/app-CarnationLetter.html. Read it. The letter's second reason is that the top and bottom must be made of thicker metal than the side. Model that: suppose the metal for the top and bottom costs k times as much per square unit as the metal for the side, where $k > 1$, so the cost of a can is proportional to

$$C = k \cdot 2\pi r^2 + 2\pi r h.$$

Find the radius that minimizes C, and show that at that radius the ratio of height to diameter is exactly k. Does this push the best can toward being taller or shorter than the answer in (a)? Is that the direction the letter says?

(c) Choose *two* of the letter's other reasons. For each, say in a sentence or two which way it pushes the shape of the can (taller and thinner, or shorter and wider, or neither), and whether you think it could be built into a function to minimize the way reason two was in (b). If yes, say roughly what you would add to C; if no, say what makes it hard to put into a formula.

(d) The optimization in (a) is correct mathematics, and the cans on a grocery shelf are not shaped like its answer. In a short paragraph, say what that does and does not tell you about the usefulness of optimization.

4. Examples to order, and a broken argument

(a) Give a function f that has a critical number at $x = 0$ because $f'(0)$ is *undefined* (not because it is 0), and that has neither a local maximum nor a local minimum at 0. Explain how you know both things.

(b) Give a function that is differentiable on the open interval $(0, 1)$ and has neither a global maximum nor a global minimum on $(0, 1)$. Explain why, and then explain why the same function *does* have both on the closed interval $[0, 1]$.

(c) A student submits the following.

> **Claim.** If $f''(c) = 0$, then f has an inflection point at $x = c$.
> **Argument.** Since $f''(c) = 0$, the second derivative is neither positive nor negative at c. So at c the graph of f is neither concave up nor concave down. So the concavity of f changes at c, which is what an inflection point is.

Identify the *first* sentence of the argument that is wrong, and say precisely what is wrong with it. Then give a specific function showing the claim is false, and check your example. Finally, state a correct claim of the same shape — a condition on f'' that *does* guarantee an inflection point at c — and say why it is true.