

- Goals
- Exponential growth & decay
 - Newton's law of cooling

Let $y(t)$ = size of population, varying over time

Suppose $y'(t) = k y(t)$ for some constant k , i.e.

"rate of growth is proportional to size of pop'n"

$$\text{Then } y' = ky \Rightarrow \frac{y'}{y} = k \Rightarrow \int \frac{y'}{y} dt = \int k dt$$

separable!

$u = y \quad du = y' dt$

$$\Rightarrow \log(y) = kt + c$$

$$\Rightarrow y = e^{kt+c} = e^c e^{kt} \quad \text{[exponentiate]}$$

$$\Rightarrow y = ae^{kt} \quad (a = e^c)$$

Setting $t=0$, we see $y(0) = ae^0 = a$, so $a = \text{initial pop'n}$.

Q If $y = ae^{kt}$, at what time t has the pop'n doubled? ($k > 0$)

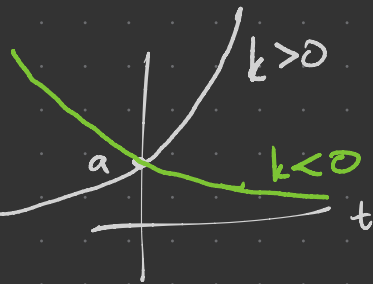
A Want t such that $y(t) = 2a$

$$ae^{kt} = 2a$$

$$e^{kt} = 2 \quad (a \neq 0)$$

$$kt = \log(2)$$

$$t = \frac{\log(2)}{k} \quad (\text{assuming } k \neq 0).$$



Newton's law of cooling

"carrying capacity"

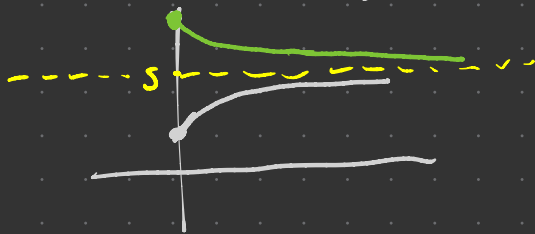
Now suppose $y' = r(S - y)$ for r, S positive constants.

Q When is the pop'n increasing? decreasing?

A y increasing $\Leftrightarrow y' > 0 \Leftrightarrow S - y > 0 \Leftrightarrow y < S$

y decreasing $\Leftrightarrow y > S$.

Q What is the long-term behavior of the pop'n?



Let's solve $y' = r(5-y)$:

$$\Rightarrow \frac{y'}{5-y} = r$$

separable!

$$\Rightarrow \int \frac{y'}{5-y} dt = \int r dt$$

$$\text{RHS} = rt + c$$

$$[a \log(x) = \log(x^a)]$$

$$\text{LHS} = -\int \frac{du}{u} = -\log(u) = -\log(5-y) = \log\left(\frac{1}{5-y}\right)$$

$u=5-y, du=-y' dt$

Thus $\log\left(\frac{1}{S-y}\right) = rt + c$

$$\Rightarrow \frac{1}{S-y} = e^{rt+c} = a e^{rt} \quad \begin{array}{l} \text{[exponentiate]} \\ [a = e^c] \end{array}$$

$$\Rightarrow S-y = \frac{1}{a e^{rt}}$$

$$\Rightarrow y = S - \frac{1}{a} e^{-rt}$$

What is the relation b/w $I = y(0)$ and this?

$$I = y(0) = S - \frac{1}{a} \Rightarrow a = \frac{1}{S-I}.$$

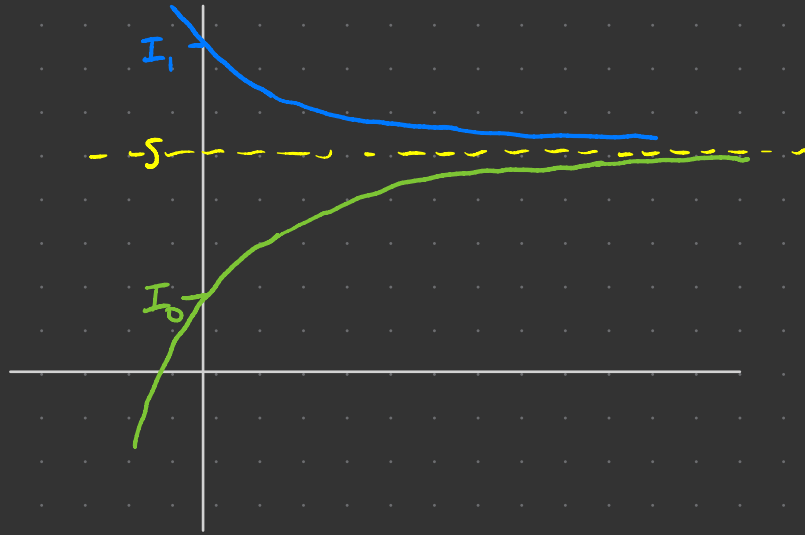
Thus

$$y = S + (I - S)e^{-rt}$$

for I the initial pop'n.

Note $y \rightarrow S$ as $t \rightarrow \infty$. Call S the carrying capacity.

Remark Could also think of S as room temperature,
 y the temperature of the liquid in a cup.



Q What does r control?

Problem You put a turkey in a 350°F oven initially at 40°F . After 10 minutes, the internal turkey temperature is 43°F . How long until the turkey is ready (165°F)?

$$43 = y(10) = 350 + (40 - 350)e^{-r \cdot 10}$$

use this to find r .