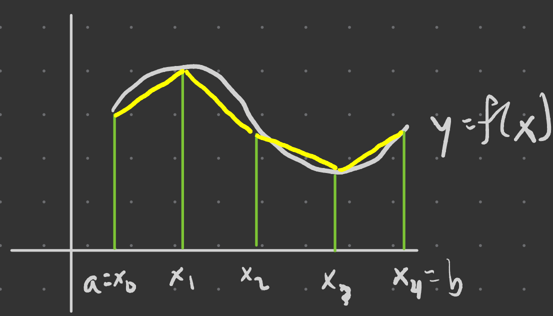
24. XI. 25

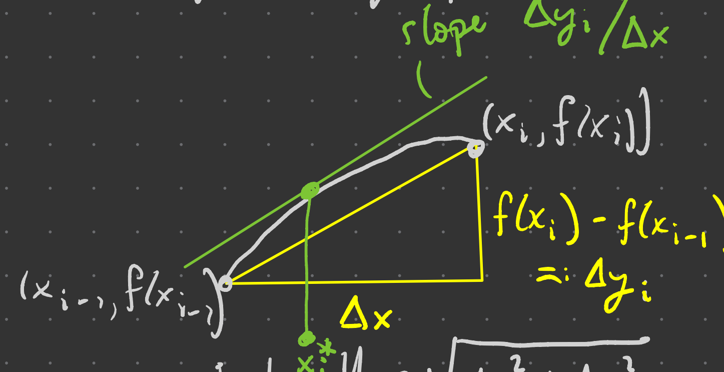
Goals

- Arclength of graphs
- Surface area of solids of revolution

Let $f : [a,b] \longrightarrow \mathbb{R}$ be differentiable.

Approximate the arclength of $y = f(x)$ by adding up secant lengths:





$$\text{secant length} = \sqrt{\Delta x^2 + \Delta y_i^2}$$

$$= \Delta x \sqrt{1 + (\Delta y_i / \Delta x)^2}$$

By MVT, there exists $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = \dfrac{\Delta y_i}{\Delta x}$,

so secant length is $\Delta x \sqrt{1 + f'(x_i^*)^2}$. Hence

$$\text{arclength} \approx \sum_{i=1}^{n} \sqrt{1 + f'(x_i^*)^2}\, \Delta x$$

$$\Rightarrow \text{arclength} = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + f'(x_i^*)^2}\, \Delta x$$

$$= \int_a^b \sqrt{1 + f'(x)^2}\, dx,$$

E.g. $f(x) = 2\sqrt{x^3}$ has arclength over $[0,1]$ given by

$$\int_0^1 \sqrt{1 + f'(x)^2}\, dx = \int_0^1 \sqrt{1 + (3x^{1/2})^2}\, dx$$

$$= \int_0^1 \sqrt{1+9x}\, dx$$

$u = 1+9x \quad du = 9dx$

$$= \int_1^{10} \frac{1}{9} u^{1/2}\, du$$

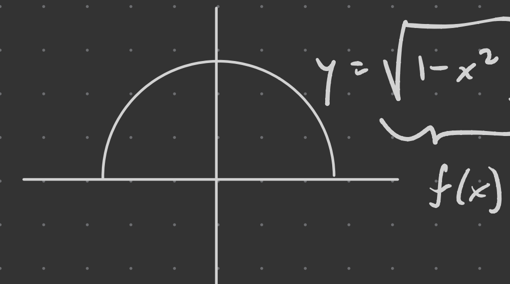
$$= \frac{1}{9} \left. \frac{u^{3/2}}{3/2} \right|_1^{10}$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \approx 2.268$$

Arclength integrals can be extremely hard to evaluate!

Nonetheless...

Q Set up — but do not evaluate! — an integral for the arclength of the unit semicircle.



$$\text{arclength} = \int_a^b \sqrt{1+f'(x)^2}\, dx$$

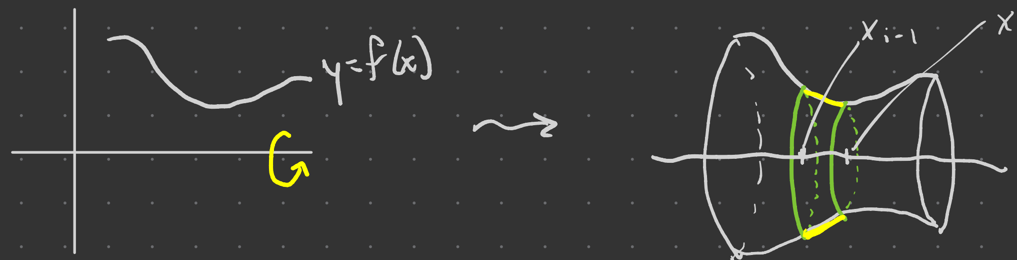
$$f'(x) = \frac{d}{dx}\left[(1-x^2)^{1/2} \right]$$

$$= \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x)$$

$$= \frac{-x}{\sqrt{1-x^2}} \implies f'(x)^2 = \frac{x^2}{1-x^2}$$

$$\text{arclength} = \int_{-1}^{1} \sqrt{1+\frac{x^2}{1-x^2}}\, dx = \int_{-1}^{1} \sqrt{\frac{1}{1-x^2}}\, dx = \pi$$

Surface area



Frustrum of a cone:

has lateral area $\pi(r_1+r_2)\ell$

(derivation in book)

The above slice has $r_1=f(x_{i-1})$, $r_2=f(x_i)$, $\ell=\Delta x\sqrt{1+\left(\frac{\Delta y_i}{\Delta x}\right)^2}$

$$=\Delta x\sqrt{1+f'(x_i^*)^2}$$

from MVT + arclength work

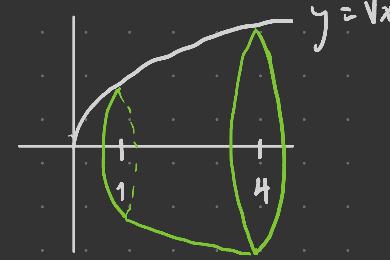
By IVT, there is also $x_i^{**} \in [x_{i-1}, x_i]$ with $f(x_i^{**}) = \frac{1}{2}(f(x_{i-1}) + f(x_i))$

$\Rightarrow$ slice area $2\pi f(x_i^{**})\sqrt{1+f'(x_i^*)^2}\ \Delta x$.

Thus surface area $\approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\sqrt{1+f'(x_i^*)^2}\ \Delta x$

and surface area $= \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\, dx$

E.g. For $f(x) = \sqrt{x}$, the surface area of



is $\int_1^4 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\, dx$

$= \int_1^4 2\pi\sqrt{x+\frac{1}{4}}\, dx$

$u = x + \frac{1}{4} \quad du = dx$

$$= \int_{5/4}^{17/4} 2\pi u^{1/2}\, du$$

$$= 2\pi \left(\frac{2}{3} u^{3/2}\right)\Big|_{5/4}^{17/4} = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \approx 30.846.$$

Q What is the surface area of the unit sphere?

Proceed by rotating $y = \sqrt{1-x^2}$, $-1 \le x \le 1$ about the x-axis.

$$SA = \int_a^b 2\pi f(x) \underbrace{\sqrt{1 + f'(x)^2}}_{= \sqrt{\frac{1}{1-x^2}}}\, dx$$

$$f(x) = \sqrt{1-x^2}$$

$$f'(x) = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$f'(x)^2 = \frac{x^2}{1-x^2}$$

$$SA = \int_{-1}^{1} 2\pi \sqrt{1-x^2} \sqrt{1 + \frac{x^2}{1-x^2}}\, dx$$

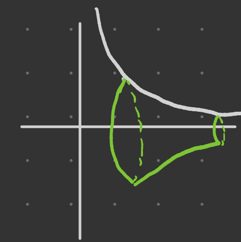
$$= \int_{-1}^{1} 2\pi \sqrt{1-x^2+x^2}\, dx$$

$$= \int_{-1}^{1} 2\pi\, dx = 2\pi x \Big|_{-1}^{1} = 4\pi \text{ units}^2$$

this works for radius r sphere as well — get $SA = 4\pi r^2$

Gabriel's horn

Rotate $y = \frac{1}{x}$, $1 \le x \le a$ about x-axis:



$$\text{volume} = \int_1^a \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^a x^{-2}\, dx$$

$$= \pi \left. \frac{x^{-1}}{-1} \right|_1^a = \pi\left(1 - \frac{1}{a}\right).$$

$$\text{surface area} = \int_1^a 2\pi \frac{1}{x} \sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\, dx$$

$$= 2\pi \int_1^a \frac{1}{x} \underbrace{\sqrt{1 + x^{-4}}}_{>1}\, dx$$

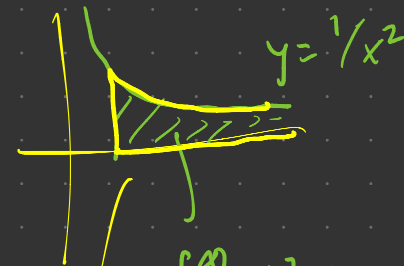
$$> 2\pi \int_1^a \frac{1}{x}\,dx = 2\pi \log x \Big|_1^a = 2\pi \log(a).$$

Cool fact $\lim_{a\to\infty} (\text{volume}) = \pi < \infty$.

Disturbing fact $\lim_{a\to\infty} (SA) > \lim_{a\to\infty} 2\pi \log(a) = \infty$

A finite volume of paint fills the horn but its surface area is infinite?

- throat of horn gets arbitrarily small — eventually a molecule won't fit
- can fit ∞ into finite spaces



perimeter
infinite

$$\int_1^\infty x^{-2}\,dx = \left.\frac{x^{-1}}{-1}\right|_1^\infty$$

$$= 1 < \infty$$