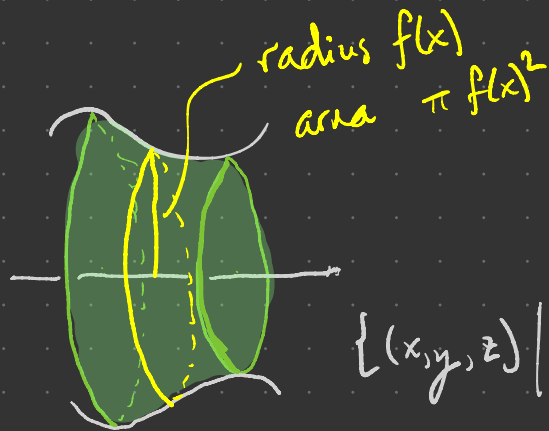
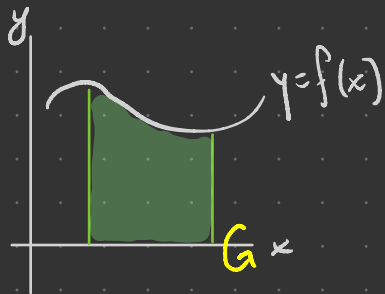


Solids of revolution

$$V = \int_{-1}^1 \sqrt{3}(1-x^2) dx = \dots$$

Take a region in the plane and rotate it about the x -axis:

E.g.

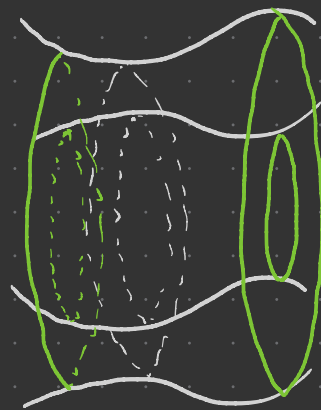
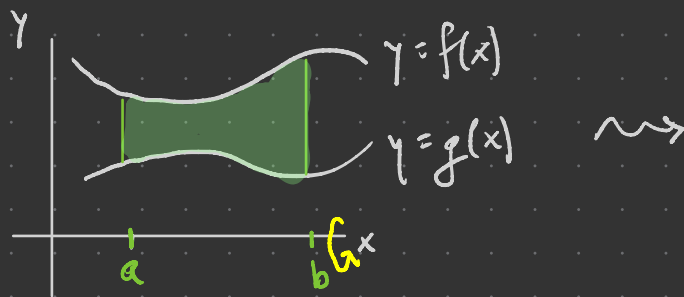


$$\{(x, y, z) \mid a \leq x \leq b, y^2 + z^2 \leq f(x)^2\}$$

$$V = \int_a^b \pi f(x)^2 dx$$

"disk method"

E.g.



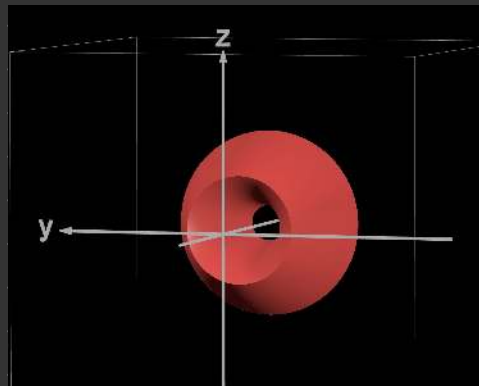
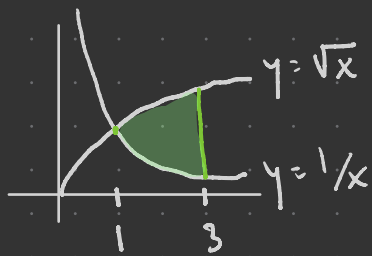
$$V = \int_a^b \pi (f(x)^2 - g(x)^2) dx$$

"washer method"



$$= \pi (R^2 - r^2)$$

E.g. Find the volume of a solid of revolution formed by revolving the region bounded by the graphs $y = \sqrt{x}$ and $y = \frac{1}{x}$ over the interval $[1, 3]$ around the x -axis.



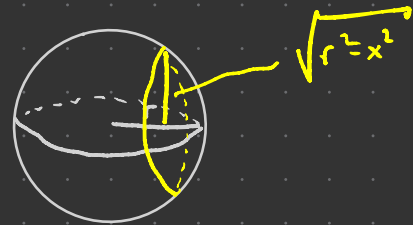
$$V = \int_1^3 \pi \left(\sqrt{x}^2 - \left(\frac{1}{x} \right)^2 \right) dx = \pi \int_1^3 (x - x^{-2}) dx$$

$$\begin{aligned}
 &= \pi \left(\frac{1}{2}x^2 - \frac{x^{-1}}{-1} \right) \Big|_1^3 = \pi \left(\frac{1}{2}x^2 + \frac{1}{x} \right) \Big|_1^3 \\
 &= \pi \left(\frac{9}{2} + \frac{1}{3} - \left(\frac{1}{2} + 1 \right) \right) \\
 &= \pi \left(3 + \frac{1}{3} \right)
 \end{aligned}$$

$$V = \frac{10\pi}{3}$$

Problem Use the disk method to compute the volume of a ball of radius r .

Hint: rotate  around x -axis.



$$V = \int_{-r}^r \pi \cdot \sqrt{r^2 - x^2}^2 dx = \pi \int_{-r}^r (r^2 - x^2) dx = \pi \left(\left(r^2 x - \frac{x^3}{3} \right) \Big|_{-r}^r \right)$$

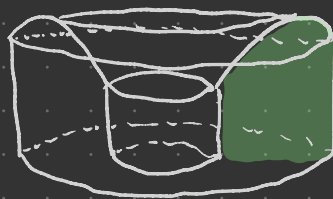
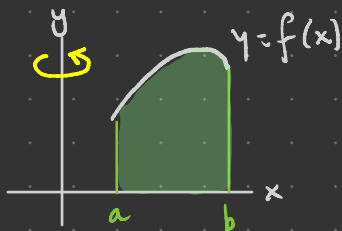
$$V = \frac{4}{3} \pi r^3$$

$$= \pi \left(\underline{r^3} - \frac{r^3}{3} - \left(\underline{-r^3} + \frac{r^3}{3} \right) \right)$$

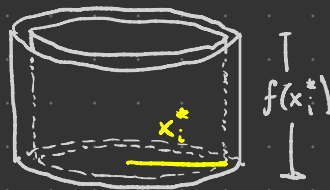
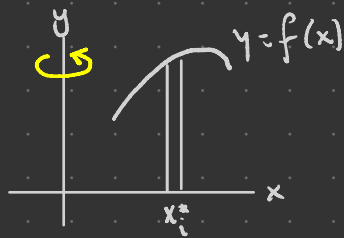
$$= \pi \left(2r^3 - \frac{2r^3}{3} \right)$$

Cylindrical Shells

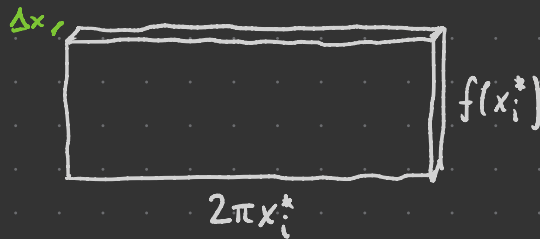
What if we rotate $\{(x, y) \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$ around the y -axis?



Slice into "shells":



unroll
↓



$$\text{Thus } V \approx \sum_{i=1}^n 2\pi x_i^* f(x_i^*) \Delta x$$

$$\text{and } V = \lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi x_i^* f(x_i^*) \Delta x$$

$$\Rightarrow V = \int_a^b 2\pi x f(x) dx$$

"shell method"

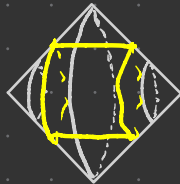
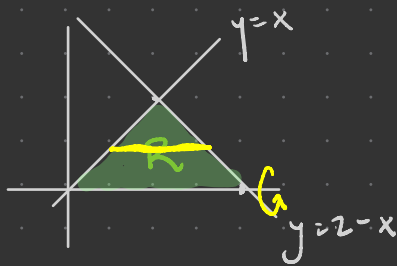
$$V_{\text{shell}} \approx 2\pi x_i^* f(x_i^*) \Delta x$$

E.g. Find the volume of the solid created by revolving
 $R = \{(x, y) \mid 1 \leq x \leq 2, 0 \leq y \leq x^2\}$ around the y -axis.



$$\begin{aligned} V &= \int_1^2 2\pi x \cdot x^2 dx = 2\pi \int_1^2 x^3 dx = 2\pi \left. \frac{x^4}{4} \right|_1^2 = \frac{\pi}{2} (16 - 1) \\ &= \boxed{\frac{15\pi}{2}} \end{aligned}$$

Problem Find the volume of the solid formed by revolving $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq x, 2-x\}$ around the x -axis.



By disks:

$$V = \int_0^1 \pi x^2 dx + \int_1^2 \pi (2-x)^2 dx$$

By shells: $x=y$ $x=2-y$

$$\begin{aligned} V &= \int_0^1 2\pi y (2-y-y) dy \\ &= \int_0^1 2\pi y (2-2y) dy \end{aligned}$$