

Goals • Integration by parts

If $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g(x) + g'(x)f(x)$.

Thus
$$\int h'(x) dx = \int (g(x)f'(x) + f(x)g'(x)) dx$$

i.e.
$$f(x)g(x) = \int g(x)f'(x) dx + \underbrace{\int f(x)g'(x) dx}_{\text{solve for this}}$$

$$\Rightarrow \int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx.$$

Setting $u = f(x)$, $v = g(x)$

$$du = f'(x) dx, \quad dv = g'(x) dx$$

this reads $\int u dv = uv - \int v du$.

Thm [Integration by parts] Let $u=f(x)$, $v=g(x)$ be functions with continuous derivatives. Then

$$\int u dv = uv - \int v du.$$

E.g. Let's compute $\int x \sin x dx$. Set $u=x$, $dv=\sin x dx$
 $\Rightarrow du=dx$, $v=-\cos x$.

By integration by parts,

$$\begin{aligned}
 \int \underbrace{x}_u \underbrace{\sin x}_{dv} dx &= uv - \int v du \\
 &= -x \cos x + \int \cos x dx \\
 &= -x \cos x + \sin x + C.
 \end{aligned}$$

Check $(-x \cos x + \sin x)' = \cancel{-\cos x} - x(-\sin x) + \cancel{\cos x}$

$$= x \sin x$$

$$\int u dv = uv - \int v du$$

Problem Evaluate $\int x e^{2x} dx$ using $u = x$, $dv = e^{2x} dx$.

$$\Rightarrow du = dx, v = \frac{1}{2} e^{2x}$$

$$\int x e^{2x} dx = \frac{1}{2} x e^{2x} - \int \frac{1}{2} e^{2x} dx$$

$$= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$$

$$= \frac{1}{4} e^{2x} (2x - 1) + C.$$

How do you choose u , dv ? Try $u =$

L - logarithmic

I - inverse trig

A - algebraic

T - trig

E - exponential

in that order.

E.g. $\int \frac{\log x}{x^3} dx = -\frac{1}{2}x^{-2}\log x + \int \frac{1}{2}x^{-3} dx$

$$u = \log x \quad dv = x^{-3} dx$$

$$du = \frac{1}{x} dx \quad v = \frac{x^{-2}}{-2}$$

$$= -\frac{\log x}{2x^2} + \frac{1}{2} \frac{x^{-2}}{-2} + C$$

$$= -\frac{\log x}{2x^2} - \frac{1}{4x^2} + C$$

$$= \frac{2\log x - 1}{4x^2} + C$$

E.g. Evaluate $\int x^2 e^{3x} dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int x e^{3x} dx$

$$u = x^2, dv = e^{3x} dx$$

$$du = 2x dx, v = \frac{1}{3} e^{3x}$$

$$\text{Now } \int x e^{3x} dx = \frac{1}{3} x e^{3x} - \frac{1}{3} \int e^{3x} dx = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C$$

$$u = x, dv = e^{3x} dx$$

$$du = dx, v = \frac{1}{3} e^{3x}$$

$$\text{so } \int x^2 e^{3x} dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \left(\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} \right) + C$$

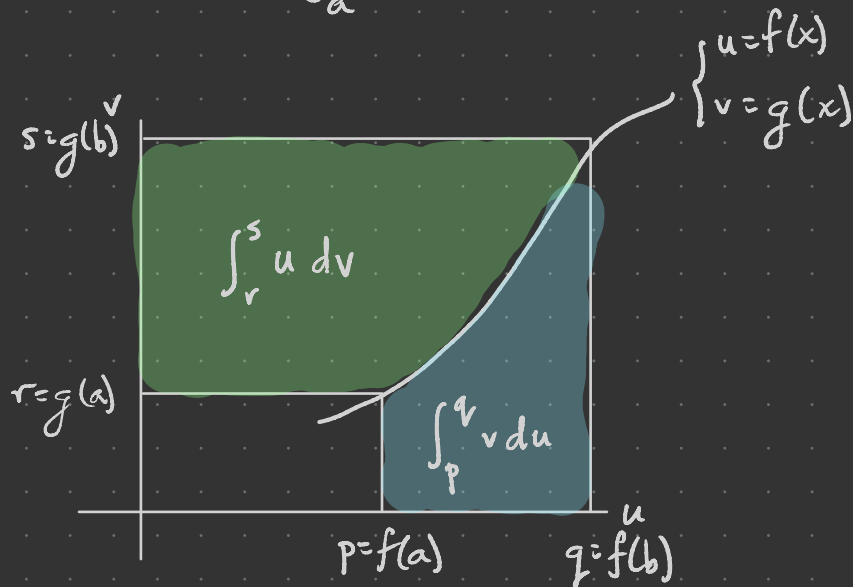
$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + C.$$

Then [Integration by parts for definite integrals]

Let $u=f(x)$, $v=g(x)$ be functions with cts derivatives on $[a,b]$.

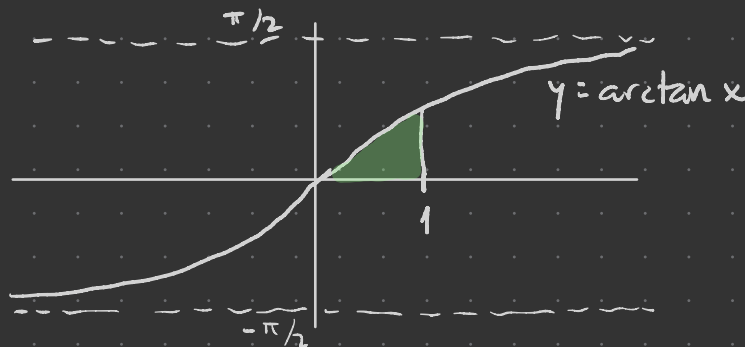
Then
$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du.$$

Pf by Picture



$$\text{Green Circle} + \text{Blue Circle} = qs - pr = uv \Big|_a^b \quad \square$$

E.g. Find the indicated area:



$$A = \int_0^1 \arctan x \, dx = x \arctan x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx$$

$$u = \arctan x \quad dv = dx \quad u = 1+x^2 \quad du = 2x \, dx$$

$$du = \frac{1}{1+x^2} \, dx \quad v = x$$

$$= \arctan(1) - \frac{1}{2} \int_1^2 \frac{du}{u}$$

$$= \frac{\pi}{4} - \frac{1}{2} \log u \Big|_1^2$$

$$= \frac{\pi}{4} - \frac{1}{2} \log(2) \approx 0.4388$$

Problem Determine $\int_0^{\pi/2} x \cos x \, dx$.