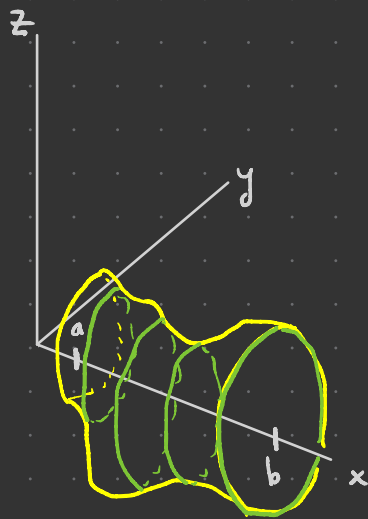


Goal • Volume = $\int_a^b$ (cross-sectional area)

• Volume of solids of revolution by disks & washers



volume $V \approx \sum_{i=1}^n A(x_i^*) \Delta x$

area of cross-section with $x = x_i^*$

and $V = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i^*) \Delta x$

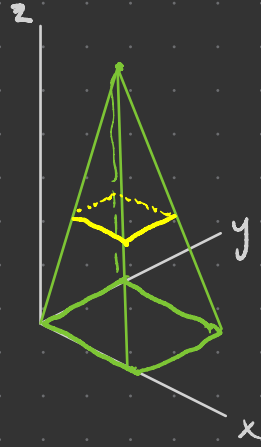
$$= \int_a^b A(x) dx$$

Slicing Method

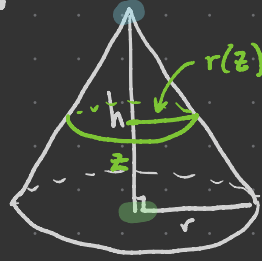
① Examine the solid and determine the shape of cross-sections (with respect to some coordinate axis, probably).

② Determine a formula for area of cross-section

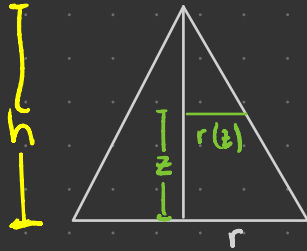
③ Integrate the area formula over the appropriate interval to get volume.



E.g. Let's find the volume of a circular cone with base radius r , height h :



$A(z) = \pi r(z)^2$ — but what is $r(z)$?



By similar triangles, $\frac{h-z}{r(z)} = \frac{h}{r}$

$$\Rightarrow r(z) = \frac{r}{h}(h-z) = r - \frac{r}{h}z$$

Thus $A(z) = \pi \left(r - \frac{r}{h}z\right)^2$ and $V = \int_0^h A(z) dz = \int_0^h \pi \left(r - \frac{r}{h}z\right)^2 dz$

bounds describe smallest & largest z -values when we slice thru shape

$$\begin{aligned} u &= r - \frac{r}{h}z \\ du &= -\frac{r}{h} dz \end{aligned}$$

$$\Rightarrow dz = -\frac{h}{r} du$$

So by substitution, $V = \pi \int_{r=u(0)}^{0=u(h)} u^2 \cdot \left(-\frac{h}{r}\right) du$

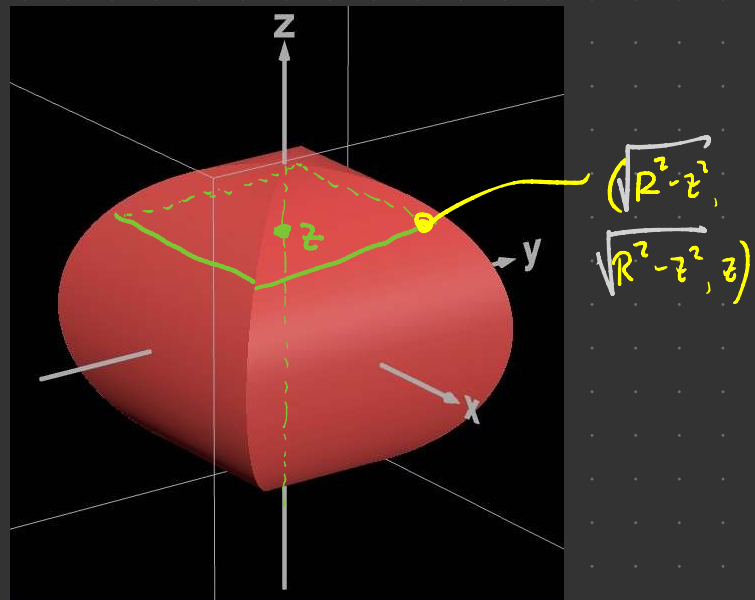
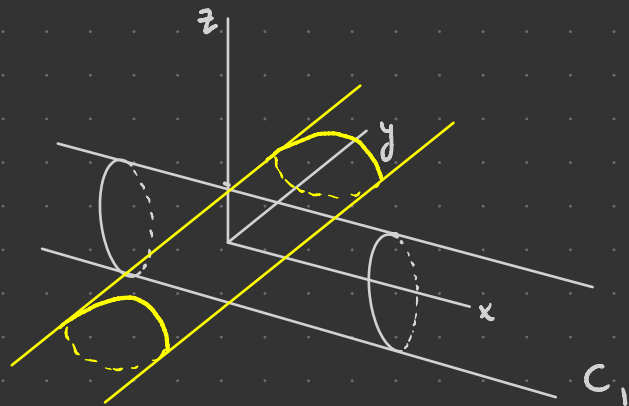
$$= -\frac{\pi h}{r} \frac{u^3}{3} \Big|_r^0$$

$$= \frac{\pi h}{r} \frac{r^3}{3}$$

$$V = \frac{1}{3} \pi r^2 h$$

E.g. Consider cylinders $C_1 = \{(x, y, z) \mid y^2 + z^2 \leq R^2\}$
 $C_2 = \{(x, y, z) \mid x^2 + z^2 \leq R^2\}$

What is the volume of $C_1 \cap C_2 = \{(x, y, z) \text{ in both } C_1 \text{ and } C_2\}$?



Q Along which axis should we slice?

A Perpendicular to z -axis so that slices are squares!

side length at height z is $\sqrt{R^2 - z^2}$ so $A(z) = 4(R^2 - z^2)$

$$\text{Thus } V = \int_{-R}^R 4(R^2 - z^2) dz = 4 \int_{-R}^R R^2 dz - 4 \int_{-R}^R z^2 dz$$

$$= 4 \int_{-R}^R (R^4 - 2R^2 z^2 + z^4) dz = 4R^2 z \Big|_{-R}^R - 4 \frac{z^3}{3} \Big|_{-R}^R$$

$$= 4 \left(R^4 z - \frac{2}{3} R^2 z^3 + \frac{1}{5} z^5 \right) \Big|_{-R}^R = 8R^3 - \frac{8}{3} R^3$$

$$= \frac{16}{3} R^3$$

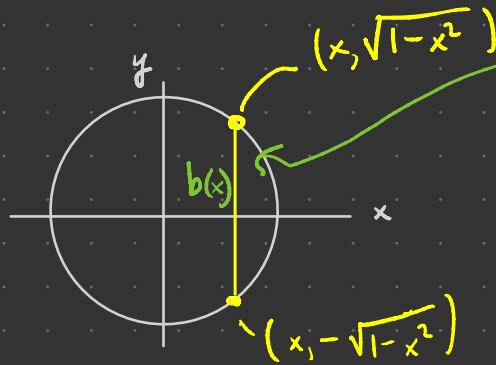
$$= 8 \left(R^5 - \frac{2}{3} R^5 + \frac{1}{5} R^5 \right)$$

$$V = \frac{16}{3} R^3$$

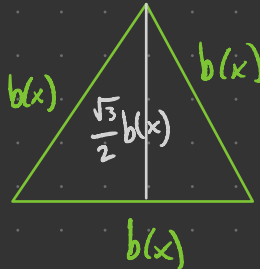
(no π !)

Problem Find the volume of the solid with base the disk of radius 1 in the xy -plane, cross-sections perpendicular to x -axis equilateral triangles.

From above:



$$b(x) = 2\sqrt{1-x^2}$$



$$A(x) = \frac{1}{2} b h$$

$$= \frac{1}{2} \frac{\sqrt{3}}{2} b(x) \cdot b(x)$$

$$= \frac{\sqrt{3}}{4} b(x)^2 = \sqrt{3} (1-x^2)$$

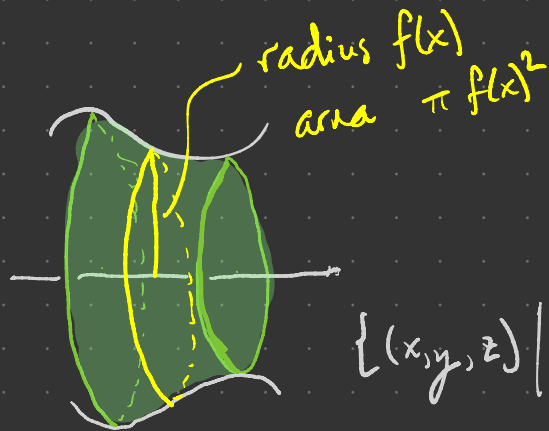
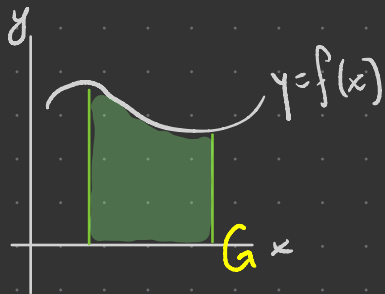


Solids of revolution

$$V = \int_{-1}^1 \sqrt{3}(1-x^2) dx = \dots$$

Take a region in the plane and rotate it about the x -axis:

E.g.

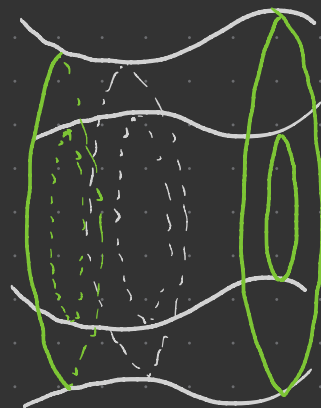
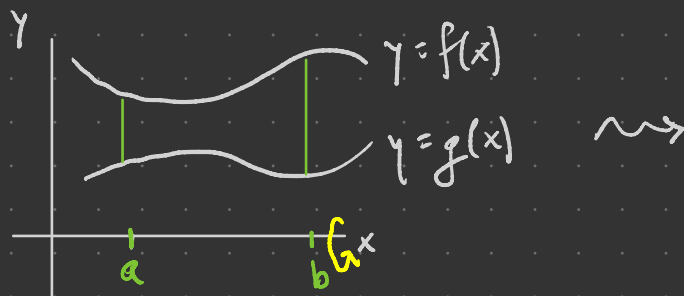


$$\{(x, y, z) \mid a \leq x \leq b, y^2 + z^2 \leq f(x)^2\}$$

$$V = \int_a^b \pi f(x)^2 dx$$

"disk method"

E.g.



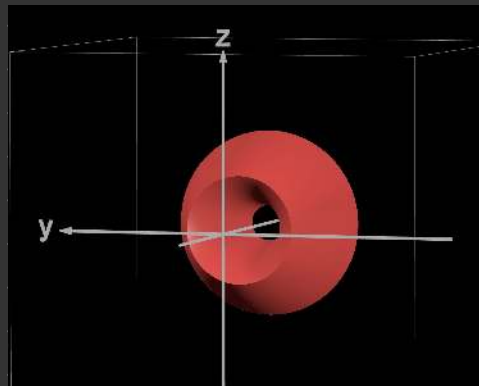
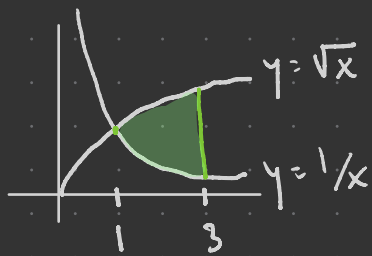
$$V = \int_a^b \pi (f(x)^2 - g(x)^2) dx$$

"washer method"



$$= \pi (R^2 - r^2)$$

E.g. Find the volume of a solid of revolution formed by revolving the region bounded by the graphs $y = \sqrt{x}$ and $y = \frac{1}{x}$ over the interval $[1, 3]$ around the x -axis.



$$V = \int_1^3 \pi \left(\sqrt{x}^2 - \left(\frac{1}{x} \right)^2 \right) dx = \pi \int_1^3 (x - x^{-2}) dx$$

$$\begin{aligned}
 &= \pi \left(\frac{1}{2} x^2 - \frac{x^{-1}}{-1} \right) \Big|_1^3 = \pi \left(\frac{1}{2} x^2 + \frac{1}{x} \right) \Big|_1^3 \\
 &= \pi \left(\frac{9}{2} + \frac{1}{3} - \left(\frac{1}{2} + 1 \right) \right) \\
 &= \pi \left(3 + \frac{1}{3} \right)
 \end{aligned}$$

$$V = \frac{10\pi}{3}$$

Problem Use the disk method to compute the volume of a ball of radius r .