24.XI.4

Goals
- FTC2
- net change

$\boxed{\textbf{FTC2}\quad \text{If } f:[a,b]\to\mathbb{R} \text{ is continuous and } F'=f, \text{ then } \int_a^b f(x)\,dx = F(b)-F(a).}$

Notation $F(x)\Big|_a^b := F(b)-F(a)$ so $\int_a^b f(x)\,dx = F(x)\Big|_a^b$.

E.g. Since $(x^3)' = 3x^2$, $\int_1^4 3x^2\,dx = x^3\Big|_1^4 = 4^3-1^3 = 63$.

or $(x^3-5)' = 3x^2$, $\int_1^4 3x^2\,dx = (x^3-5)\Big|_1^4 = (4^3-\cancel{5})-(1^3-\cancel{5})$
$= 63$

Proof of FTC2

Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a regular partition of $[a,b]$. Then

$$F(b) - F(a) = F(x_n) - F(x_0)$$

$$= F(x_n) + 0 + 0 + \cdots + 0 - F(x_0)$$

$$= \left[F(x_n) - F(x_{n-1})\right] + \left[F(x_{n-1}) - F(x_{n-2})\right] + F(x_{n-2}) - \cdots$$

$$- F(x_1) + \left[F(x_1) - F(x_0)\right]$$

$i=1$ $\quad$ $i=2$

$$F(x_1) - F(x_0) + F(x_2) - F(x_1)$$

$$+ F(x_3) - F(x_2)$$

$$+ \cdots + F(x_n) - F(x_{n-1})$$

$$= \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1})\right]$$

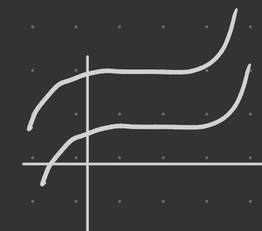
MVT: $F'(c) = \dfrac{F(b) - F(a)}{b-a}$ for some c in (a,b)

$$= \sum_{i=1}^{n} F'(c_i)(x_i - x_{i-1}) \quad \text{for some } c_i \in [x_{i-1}, x_i]$$

$$= \sum_{i=1}^{n} f(c_i)\,\Delta x \xrightarrow[n \to \infty]{} \int_a^b f(x)\,dx. \quad \square$$

When $F' = f$ we call F an <u>antiderivative</u> of f and write

$$\int f(x)\,dx := F(x) + C$$



indefinite integral of f

where C is a constant.

<u>Note</u> Antiderivatives aren't unique!

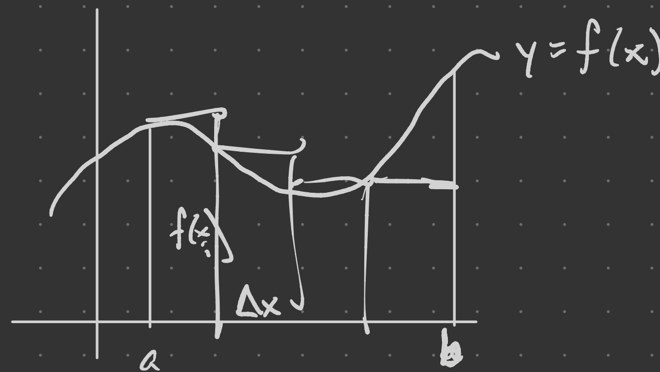
$$(x^3)' = 3x^2$$

$$(x^3 - 5)' = 3x^2$$

FTC2 says

$$\int_a^b f(x)\,dx = \left(\int f(x)\,dx \right) \Bigg|_a^b .$$

Informal justification of FTC2:



$$f(x_i) = F'(x_i)$$

$$f(x_i)\,\Delta x = F'(x_i)(x_i - x_{i-1})$$

$$\approx F(x_i) - F(x_{i-1})$$

MVT

+ telescoping sum.

Some antiderivatives

- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$

so taking log of positive

- $\int \frac{1}{x} \, dx = \log|x| + C$ for $x \neq 0$

- $\int e^x \, dx = e^x + C$

- $\int \cos x \, dx = \sin x + C$

- $\int \sin x \, dx = -\cos x + C$

- $\int \sec^2 x \, dx = \tan x + C$

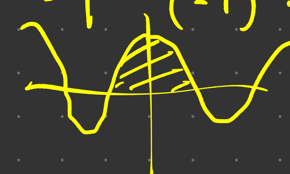
Some definite integrals

- $\int_0^1 \sqrt{x} \, dx = \frac{x^{3/2}}{3/2}\Big|_0^1 = \frac{1}{3/2} - \frac{0}{3/2}$
  $= \frac{2}{3}$

- $\int_1^3 \frac{1}{x} \, dx = \log 3 - \log 1 = \log 3$

- $\int_{-1000}^{1000} e^x \, dx = e^{1000} - e^{-1000}$

- $\int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \sin(\pi/2) - \sin(-\pi/2)$
  $= 1 - (-1) = 2$



Interpretation (Slogan) Integrating rate of change computes net change.

Indeed, $\int_a^b F'(x)\,dx = F(b) - F(a)$.

E.g. $v(t) = s'(t)$ for $s(t)$ = position at time t

$$\int_a^b v(t)\,dt = s(b) - s(a) = \text{net change in position} = \text{displacement}$$

E.g. For instance, if $v(t) = \frac{1}{2}t + 3$, then determining an antiderivative will allow us to compute displacement.

For the interval $[0, 5]$,

$$s(5) - s(0) = \int_0^5 \left(\frac{1}{2}t + 3\right) dt = \frac{1}{2}\int_0^5 t \, dt + 3\int_0^5 1 \, dt$$

$$= \frac{1}{2}\left(\frac{t^2}{2}\Big|_0^5\right) + 3 \; t\Big|_0^5$$

$$= \frac{1}{2}\left(\frac{5^2}{2} - \frac{0^2}{2}\right) + 3(5 - 0)$$

$$= \frac{25}{4} + 15 .$$

E.g. $\displaystyle\int_1^2 t \cdot (t^4 + 1) \, dt$

$$\int_a^b f(x) g(x) \, dx \neq \int_a^b f(x) \, dx \cdot \int_a^b g(x) \, dx$$

$$= \int_1^2 (t^5 + t)\,dt = \int_1^2 t^5\,dt + \int_1^2 t\,dt$$

$$= \frac{t^6}{6}\Big|_1^2 + \frac{t^2}{2}\Big|_1^2 = \frac{2^6}{6} - \frac{1}{6} + \frac{4}{2} - \frac{1}{2} = \cdots$$

E.g.: $$\int_2^3 \frac{x^3 - x}{\sqrt{x}}\,dx = \int_2^3 x^{-1/2}(x^3 - x)\,dx$$

$$= \int_2^3 (x^{2.5} - x^{1/2})\,dx = \int_2^3 x^{2.5}\,dx - \int_2^3 x^{1/2}\,dx$$

$$= \cdots$$