

Goals · Evaluate indefinite and definite integrals
via substitution

Recall $\underbrace{\int f(x) dx = F(x) + C}_{\text{indefinite integral of } f}$ means that $F'(x) = f(x)$.

Differentiation rules $\rightsquigarrow$ Antidifferentiation rules

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

linearity $\left\{ \begin{array}{l} \frac{d}{dx} (f+g) = \frac{df}{dx} + \frac{dg}{dx} \\ \frac{d}{dx} [cf] = c \frac{df}{dx} \end{array} \right.$

$$\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$$

$$\int cf(x) dx = c \int f(x) dx$$

chain rule:

$$(f \circ g)'(x) = f'(g(x)) g'(x)$$

substitution

??

product rule

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

integration by parts

??

Thm [Substitution for indefinite integrals] Suppose f, g' cts.

$$\text{If } F' = f, \text{ then } \int f(g(x)) g'(x) dx = F(g(x)) + C.$$

By setting $u = g(x)$, $du = g'(x) dx$, we can also write this
as
$$\int f(g(x)) g'(x) dx = \int f(u) du.$$

Pf By the chain rule,

$$\begin{aligned}\frac{d}{dx} F(g(x)) &= F'(g(x)) g'(x) \\ &= f(g(x)) g'(x).\end{aligned}$$

Integrating,

$$F(g(x)) + C = \int f(g(x)) g'(x) dx. \quad \square$$

E.g.

$$\begin{aligned}\int \underbrace{(x^2 - 3)^3}_u \cdot \underbrace{2x dx}_{du} &= \int u^3 du \\ &= \frac{u^4}{4} + C \\ &= \frac{(x^2 - 3)^4}{4} + C.\end{aligned}$$

$$\begin{aligned}\int \underbrace{f(g(x))}_u \underbrace{g'(x) dx}_{du} &= F(g(x)) + C \\ &= F(u) + C \\ \text{for } F' &= f\end{aligned}$$

Substitution Strategy

- (1) Look for $g(x)$ in integrand such that $\tilde{g}'(x)$ is also in the integrand.
- (2) Substitute $u=g(x)$, $du=g'(x)dx$ into the integral
- (3) Integrate with respect to variable u .
- (4) Rewrite the result in terms of x

E.g. $\int 3x^2(x^3-3)^2 dx = \int u^2 du$

$u = x^3 - 3$
 $du = 3x^2 dx$

$$= \frac{u^3}{3} + C$$
$$= \frac{(x^3-3)^3}{3} + C$$

Check

$$\frac{d}{dx} \frac{(x^3-3)^3}{3}$$
$$= \cancel{3} \frac{(x^3-3)^2}{\cancel{3}} \cdot 3x^2$$
$$= (x^3-3)^2 \cdot 3x^2 \quad \checkmark$$

E.g. $\int z \sqrt{z^2 - 5} \, dz = ?$

$$\text{Set } u = z^2 - 5, \quad du = 2z \, dz \Rightarrow z \, dz = \frac{1}{2} du.$$

$$\text{Thus } \int z \sqrt{z^2 - 5} \, dz = \int \frac{1}{2} u^{1/2} \, du$$

$$= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3} u^{3/2} + C$$

$$= \frac{1}{3} (z^2 - 5)^{3/2} + C.$$

Problems (1) Find $\int x^2 (x^3 + 5)^9 \, dx$

(2) Find $\int \frac{\cos t}{\sin^2 t} \, dt$

(1) Set $u = x^3 + 5$, so $du = 3x^2 dx \Rightarrow \frac{1}{3} du = x^2 dx$.

By substitution,

$$\int x^2 (x^3 + 5)^9 dx = \int \frac{1}{3} u^9 du$$

$$= \frac{1}{3} \frac{u^{10}}{10} + C$$

$$= \frac{1}{30} (x^3 + 5)^{10} + C$$

(2) $\int \frac{\cos t}{\sin^2 t} dt = \int \cos t \cdot \frac{1}{\sin^2 t} dt = \int \frac{1}{u^2} du = \int u^{-2} du$

$$u = \sin t$$

$$du = \cos t dt$$

$$= \frac{u^{-1}}{-1} + C = \frac{-1}{\sin t} + C$$

E.g. $\int \frac{x}{\sqrt{x-1}} dx = \int \frac{u+1}{\sqrt{u}} du = \int \frac{u+1}{u^{1/2}} du$ $\left[= -\csc t + C \right]$

$u = x-1 \Rightarrow x = u+1$
 $du = dx$

$$\frac{u+1}{\sqrt{u}} = \frac{u}{\sqrt{u}} + \frac{1}{\sqrt{u}}$$

$$= \frac{u}{u^{1/2}} + \frac{1}{u^{1/2}}$$

$$= u^{1/2} + u^{-1/2}$$

$$= \int (u^{1/2} + u^{-1/2}) du = \frac{u^{3/2}}{3/2} + \frac{u^{1/2}}{1/2} + C$$

$$= \frac{2}{3} u^{3/2} + 2u^{1/2} + C$$

$$= \frac{2}{3} (x-1)^{3/2} + 2(x-1)^{1/2} + C$$

$$= \frac{2}{3} (x-1)^{1/2} ((x-1) + 3) + C$$

$$= \frac{2}{3} (x-1)^{1/2} (x+2) + C$$

Problem Check this answer!

$$\begin{aligned}\frac{d}{dx} \left[\frac{2}{3} (x-1)^{3/2} + 2 (x-1)^{1/2} \right] &= (x-1)^{1/2} + (x-1)^{-1/2} \\ &= \sqrt{x-1} + \frac{1}{\sqrt{x-1}} = \frac{x-1+1}{\sqrt{x-1}} \\ &= \frac{x}{\sqrt{x-1}} \quad \checkmark\end{aligned}$$

Thm [Substitution for definite integrals]

Let $u=g(x)$ with g' continuous on $[a,b]$. Let f be continuous over the image of $u=g(x)$. Then

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Pf If $\int f(u) du = F(u) + C$, then

$$\int_a^b f(g(x)) g'(x) dx = F(g(x)) \Big|_{x=a}^{x=b}$$
$$= F(g(b)) - F(g(a))$$

$$= \int_{g(a)}^{g(b)} f(u) du \quad \square$$

E.g.

$$\int_{-1}^0 y (2y^2 - 3)^5 dy = \int_{2(-1)^2 - 3}^{2(0)^2 - 3} \frac{1}{4} u^5 du = \int_{-1}^{-3} \frac{1}{4} u^5 du$$

$u = 2y^2 - 3$
 $du = 4y dy$
 $y dy = \frac{1}{4} du$

$2(0)^2 - 3 = u(0)$
 $2(-1)^2 - 3$
 $u(-1)$

$$= \frac{1}{4} \frac{u^6}{6} \bigg|_{u=-1}^{u=-3}$$

$$= \frac{1}{24} \left((-3)^6 - (-1)^6 \right)$$

$$= \frac{91}{3}.$$

Problem Evaluate $\int_0^1 x^2 \cos\left(\frac{\pi}{2} x^3\right) dx$.

E.g. Find $\int_0^{\pi/2} \cos^2 \theta \, d\theta$.

First, $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$, so

$$\int_0^{\pi/2} \cos^2 \theta \, d\theta = \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta + \frac{1}{2} \int_0^{\pi/2} \cos 2\theta \, d\theta$$

$$= \frac{\pi}{4} + \frac{1}{2} I.$$

Now using $u = 2\theta$, $du = 2d\theta$,

$$\begin{aligned} I &= \int_0^\pi \frac{1}{2} \cos u \, du = \frac{1}{2} \sin u \Big|_{u=0}^{u=\pi} \\ &= \frac{1}{2} (\sin \pi - \sin 0) \\ &= 0. \end{aligned}$$

$$\begin{aligned} \text{Thus } \int_0^{\pi/2} \cos^2 \theta \, d\theta &= \frac{\pi}{4} + \frac{1}{2} I \\ &= \frac{\pi}{4}. \end{aligned}$$

$$\log(x) := \int_1^x \frac{1}{t} dt$$

$$\log(ab) = \int_1^{ab} \frac{1}{t} dt = \int_1^a \frac{1}{t} dt + \int_a^{ab} \frac{1}{t} dt$$

$$\begin{aligned} u &= \frac{1}{a}t & du &= \frac{1}{a} dt \\ \Rightarrow t &= au & dt &= a du \end{aligned}$$

$$= \log(a) + \int_1^b \frac{1}{au} a du$$

$$= \log(a) + \log(b)$$