

- Goals
- Mean value theorem for integrals
 - State FTC1 & understand why it's true
 - Use FTC1 to evaluate derivatives of accumulation functions

Mean value theorem: $f: [a, b] \rightarrow \mathbb{R}$ cts on $[a, b]$, diff'l on (a, b) .

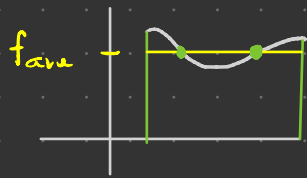
Then there exists c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Slogan Instantaneous rate of change at some point
is equal to avg rate of change.

Mean value theorem for integrals If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then there exists c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$$



Slogan A continuous function takes its average value at least once.

Proof By the extreme value theorem, f attains its min and max values m, M on $[a, b]$. Then $m \leq f(x) \leq M$ on $[a, b]$, so

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

$$\Rightarrow m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M.$$



Since $\frac{1}{b-a} \int_a^b f(x) dx$ is between m, M

+ f takes the values m, M

+ f continuous

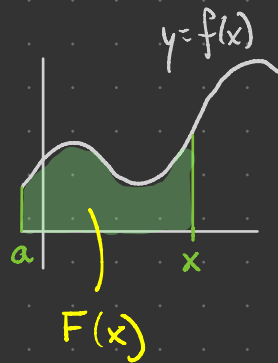
the intermediate value theorem implies there is some c for which

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx. \quad \square$$

FTC 1 If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, and

$$F(x) := \int_a^x f(t) dt, \quad \sim \text{accumulative function of } f$$

then $F'(x) = f(x)$.



Call $F(x) = \int_a^x f(t) dt$ an accumulation function.

Slogan Accumulation function of f is an antiderivative of f .

antiderivative of f

is a function F such that $F' = f$

Pf We have

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left(\int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right)$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt$$

average of f over $[x, x+h]$

$$\left[\int_a^b = \int_a^c + \int_c^b \right]$$

By MVT 4, there is some $c \in [x, x+h]$ such that

$$f(c) = \frac{1}{h} \int_x^{x+h} f(t) dt.$$

As $h \rightarrow 0$, $c \rightarrow x$. Since f is continuous,

$$\lim_{h \rightarrow 0} f(c) = \lim_{c \rightarrow x} f(c) = f(x).$$

$$\text{Thus } F'(x) = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt$$

$$= \lim_{h \rightarrow 0} f(c)$$

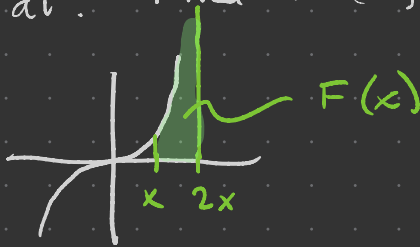
$$= f(x) \text{ as desired. } \square$$

E.g. If $g(x) = \int_1^x \frac{1}{t^3+1} dt$, then $g'(x) = \frac{1}{x^3+1}$

E.g. If $F(x) = \int_1^{x^3} \cos t \, dt$, then we can use the chain rule to compute $F'(x)$. Let $u(x) = x^3$. Then $F(x) = \int_1^{u(x)} \cos t \, dt$

$$\begin{aligned} \text{and } F'(x) &= \cos(u(x)) \cdot u'(x) \\ &= \cos(x^3) \cdot 3x^2. \end{aligned}$$

Problem Let $F(x) = \int_x^{2x} t^3 \, dt$. Find $F'(x)$.



Answer $F(x) = \int_0^{2x} t^3 dt - \int_0^x t^3 dt$

so $F'(x) = 16x^3 - x^3 = 15x^3$

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? ←

$$\int_0^{2x} t^3 dt = g(2x) \quad \text{for} \quad g(x) = \int_0^x t^3 dt$$

thus  $\frac{d}{dx} \int_0^{2x} t^3 dt = g'(2x) \cdot (2x)'$   
 $= (2x)^3 \cdot 2 = 16x^3$

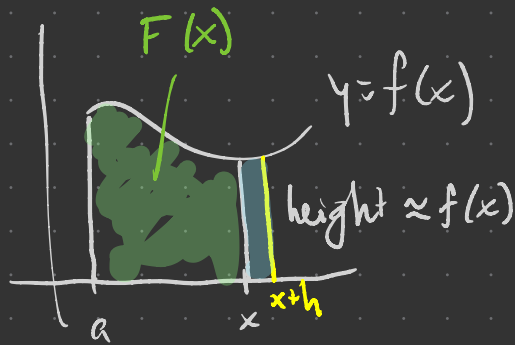
Note If  $G(x) = \int_a^{u(x)} f(t) dt$ , then

$$G'(x) = f(u(x)) \cdot u'(x).$$

(via chain rule).

(1) Informal proof of FTC1

(2) Continuity hypothesis

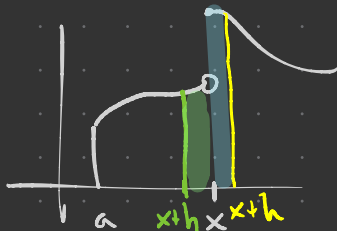


$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$$

blue area

$$\frac{F(x+h) - F(x)}{h} \approx \frac{h \cdot f(x)}{h} = f(x)$$

What if  $f$  has discontinuity?



$$\text{so } \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \text{ DNE}$$