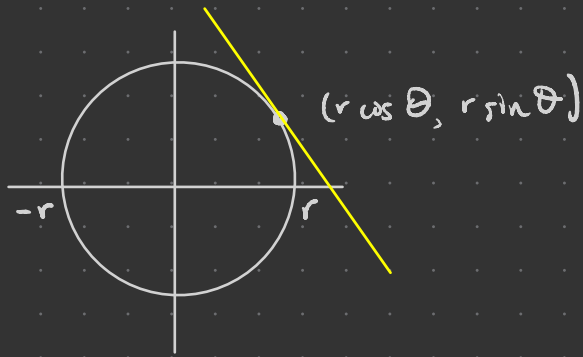


Implicit Differentiation

Motivation Suppose we want to find tangent lines to a radius r circle centered at the origin.



This circle has equation

$$x^2 + y^2 = r^2$$

$$\text{so } y = \begin{cases} \sqrt{r^2 - x^2} \\ -\sqrt{r^2 - x^2} \end{cases}$$

 Multi-valued
so not a
function!



Treat y as a "local" function of x
to find derivatives.

① Differentiate both sides of the equation, using the chain rule to treat y as a function of x

② Solve for $\frac{dy}{dx}$:

(a) Put $\frac{dy}{dx}$ terms on left side of eq'n,
factor out $\frac{dy}{dx}$

(b) Solve for $\frac{dy}{dx}$ by dividing both sides by
other factor on left.

Eg. $\frac{d}{dx}(x^2 + y^2 = r^2)$ $\downarrow \frac{d}{dx}$ — chain rule!

$$2x + 2y \frac{dy}{dx} = 0$$

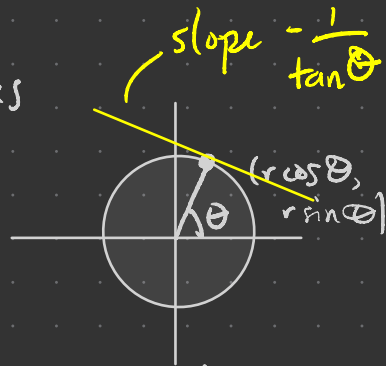
$$\Rightarrow 2y \frac{dy}{dx} = -2x \quad \text{for } y \neq 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

Depends on x and y !
Typical for implicit differentiation

So the tangent line through $(r \cos \theta, r \sin \theta)$ has

slope $\frac{-r \cos \theta}{r \sin \theta} = -\cot \theta$



E.g. Find $\frac{dy}{dx}$ for the curve given by $x^3 \sin y + y = 4x + 3$:

$$\frac{d}{dx}(x^3 \sin y) + \frac{dy}{dx} = 4$$

$$3x^2 \sin y + x^3 \cos y \frac{dy}{dx} + \frac{dy}{dx} = 4$$

$$(x^3 \cos y + 1) \frac{dy}{dx} = 4 - 3x^2 \sin y$$

$$\frac{dy}{dx} = \frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}$$

so slope of
tangent at
(0,3) is
 $\frac{4-0}{0+1} = 4$

Problem Find $\frac{dy}{dx}$ when $xy \cos(xy) + 1 = 0$.

Derivatives of logs

Recall $\log(x)$ is inverse to e^x (i.e. $\log = \log_e = \ln$)

$\log_b(x)$ is inverse to b^x

Then for $y = \log x$, $e^y = x$. Differentiating both sides,
 $e^y \frac{dy}{dx} = 1$ for $x > 0$.

$$\Rightarrow x \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \text{ for } x > 0$$

$$\text{i.e. } \boxed{\log'(x) = \frac{1}{x} \text{ for } x > 0.}$$

If $y = \log_b x$, then $b^y = x$. Applying $\log$ to both sides,

$$y \log b = \log x$$

$$y = \frac{\log x}{\log b}$$

$$\text{Thus } \boxed{\log'_b(x) = \frac{1}{x \log b}} \text{ for } x > 0.$$



If $y = b^x$, then $\log y = x \log b$. Differentiating,

$$\frac{1}{y} \frac{dy}{dx} = \log b$$

$$\Rightarrow \frac{dy}{dx} = y \log b$$

i.e. $\boxed{\frac{d}{dx} b^x = (\log b) b^x}$.

We just employed a new technique!: Logarithmic differentiation:

$$y = f(x)^{g(x)} \Rightarrow \log y = g(x) \log f(x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = g'(x) \log f(x) + \frac{g(x)}{f(x)} f'(x)$$

$$\Rightarrow \frac{dy}{dx} = f(x)^{g(x)} \cdot (g'(x) \log f(x) + \frac{g(x)}{f(x)} f'(x))$$

E.g. $y = x^x \Rightarrow \log y = x \log x$

$$\frac{1}{y} \frac{dy}{dx} = \log x + \frac{x}{x} = 1 + \log x$$

$$\frac{dy}{dx} = x^x (1 + \log x)$$