

- Goals
- Rolle's theorem
 - Mean value theorem
 - Sign of $f'(x)$ detects increasing/decreasing.

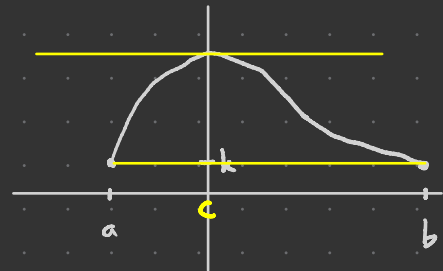
Rolle's Thm Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous, differentiable over (a, b) , and such that $f(a) = f(b)$. Then there is at least one c in (a, b) such that $f'(c) = 0$.

Pf Let $k = f(a) = f(b)$.

Case 1: $f(x) = k$ for all x in (a, b)

Then $f'(x) = 0$ for all x in (a, b) ✓

Case 2: There exists x in (a, b) such that $f(x) > k$.



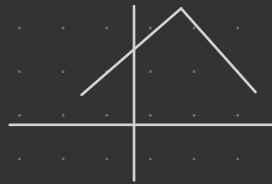
By the extreme value theorem, f has an absolute max on $[a, b]$, and this occurs at some c in (a, b) .

By Fermat's theorem, c is a critical point, so $f'(c) = 0$. ✓

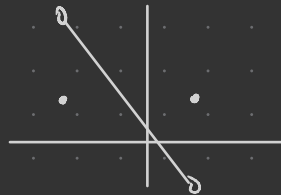
Case 3: There exists x in (a, b) such that $f(x) < k$.

Use a similar argument! □

Note Must have f diff'l for Rolle to apply:



Also must have f continuous:



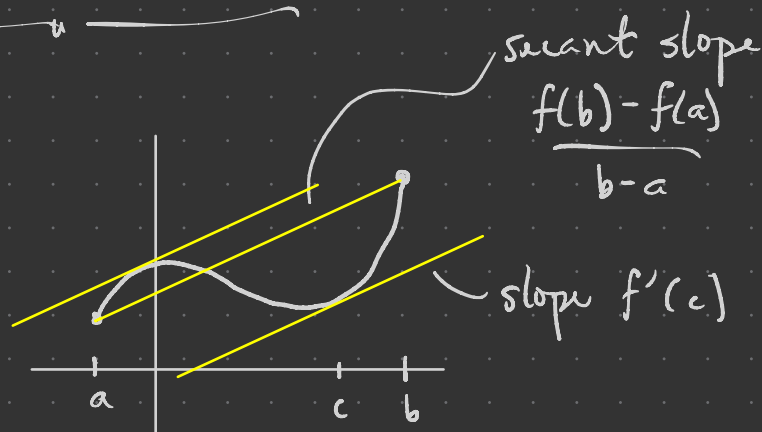
E.g. Consider $f(x) = x^3 - 9x = x(x^2 - 9) = x(x+3)(x-3)$.

Since $f(0) = f(-3) = f(3) = 0$, know f has a critical point in $(-3, 0)$ and in $(0, 3)$ by Rolle.

Check: $f'(x) = 3x^2 - 9 = 3(x^2 - 3) = 3(x + \sqrt{3})(x - \sqrt{3})$

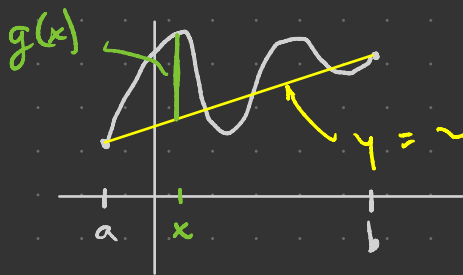
has roots at $x = \pm\sqrt{3}$ ✓

What if $f(a) \neq f(b)$?



Mean value theorem Let f be continuous over $[a, b]$,
diff'l over (a, b) . Then there exists at least one c in (a, b)
such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Pf Set $g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a} (x - a) + f(a) \right)$



Then g is continuous on $[a, b]$, diff'l on (a, b) , and

$g(a) = g(b) = 0$. By Rolle, there exists c in (a, b) with $g'(c) = 0$. But $g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$

$$\text{so } 0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$
$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a} \quad \square$$

Application If a car's average speed exceeds the speed limit, then there is at least one moment when the car's instantaneous speed exceeds the speed limit.

Three corollaries of MVT:

Cor 1 Let f be diff'l over an interval I .

If $f'(x) = 0$ for all x in I , then $f(x)$ is constant on I .

"If you don't change, you stay the same!"

Pf Suppose f is not constant and take $a < b$ with $f(a) \neq f(b)$.

Then $\frac{f(b) - f(a)}{b - a} \neq 0 \Rightarrow$ there is some c in I with $f'(c) = \frac{f(b) - f(a)}{b - a} \neq 0$,
contradicting hypothesis that $f' = 0$. $\square$

Cor 2 If f, g diff'l on an interval I and $f'(x) = g'(x)$

for all x in I , then $f(x) = g(x) + C$ for some constant C .

corollary noun

corollary kō-rō-ler-ē kār-, -le-rē, British kō-rā-lo-rē

plural **corollaries**

Synonyms of *corollary* >

1 : a proposition (see **PROPOSITION** entry 1 sense 1c) inferred immediately from a proved proposition with little or no additional proof

2 a : something that naturally follows : **RESULT**

... love was a stormy passion and jealousy its normal *corollary*.
— Ida Tread

b : something that incidentally or naturally accompanies or parallels

A *corollary* to the problem of the number of vessels to be built was that of the types of vessels to be constructed.
— Daniel Marx

Pf Apply Cor 1 to $f(x) - g(x) = h(x)$. Indeed, $h'(x) = f'(x) - g'(x) = 0 \Rightarrow h(x) = C \Rightarrow f(x) = g(x) + C$. $\square$

Cor 3 Let f be continuous on $[a, b]$, diff'l on (a, b) .

(i) If $f'(x) > 0$ for x in (a, b) , then f is increasing on $[a, b]$.

(ii) If $f'(x) < 0$ for x in (a, b) , then f is decreasing on $[a, b]$.

Pf of (i) For contradiction, suppose f not increasing on I ,

so there exist $a < b$ in I with $f(b) < f(a)$.

By MVT, there exists c in (a, b) with

$$f'(c) = \frac{f(b) - f(a)}{b - a} < 0$$

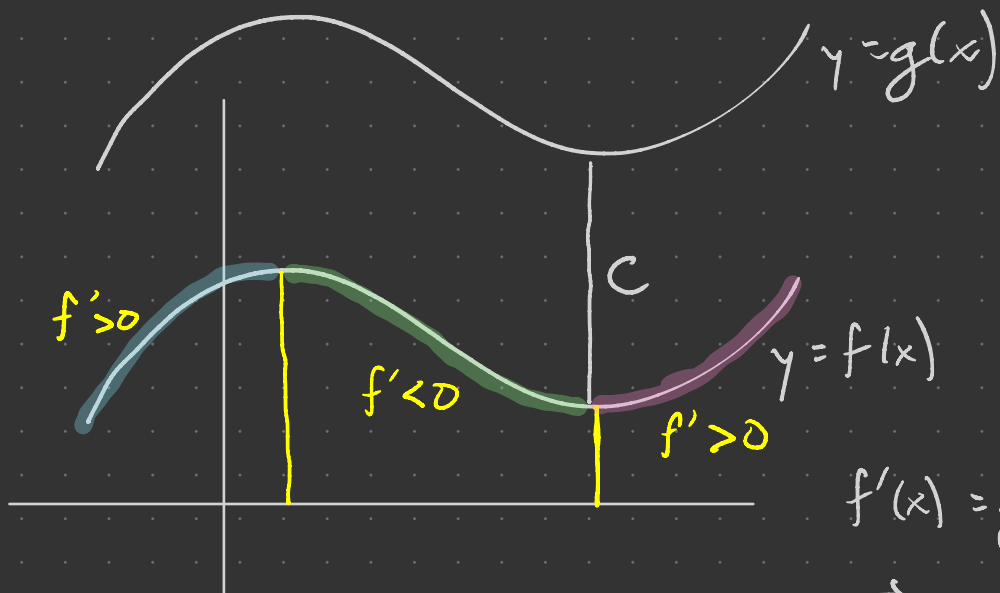
a contradiction! $\square$

Increasing:

$$c < d \Rightarrow f(c) < f(d)$$

Decreasing:

$$c < d \Rightarrow f(c) > f(d)$$



$$f'(x) = g'(x)$$

$$\Rightarrow g(x) = f(x) + C$$