

$$= e^{-1/x^2} \cdot (2x^{-3})$$

$$= \frac{2e^{-1/x^2}}{x^3}$$

Later! ☺

24. X. 2

Recall $\log_b = B^{-1}$ for $B(x) = b^x$.

Let $\log_e = \exp^{-1} = \log$ (also denoted $\ln$).

Then $(\log x)' = \frac{1}{x}$ for $x > 0$.

If By the inverse function theorem,

$$(\log_e x)' = \frac{1}{\exp(\log x)}$$

$$= \frac{1}{x} \quad \square$$

After learning about implicit differentiation, we will show

$$(\log_b x)' = \frac{1}{(\log b) x}$$

$$(b^x)' = (\log b) b^x$$

Optimization

"absolute"

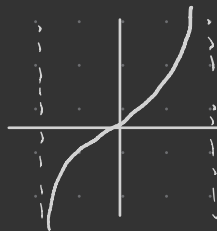
$x \in A$
"x is in A"

Let I be an interval and $f: I \rightarrow \mathbb{R}$ a function.

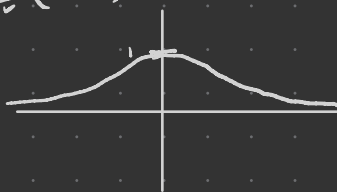
Defn We say f has a global maximum on I at c when $f(c) \geq f(x)$ for all $x \in I$. We say it has a global minimum on I at c when $f(c) \leq f(x)$ for all $x \in I$. A global max/min is called a global extremum.

E.g. • $f(x) = \tan x$ on $(-\pi/2, \pi/2)$

no global extrema



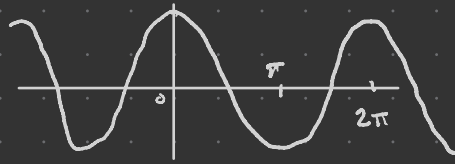
• $g(x) = \frac{1}{1+x^2}$ on $\mathbb{R} = (-\infty, \infty)$



Global max at $x=0$

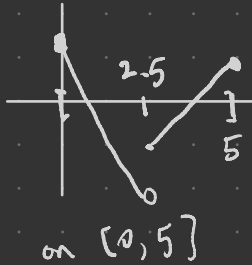
No global min.

• $h(x) = \cos x$ on $\mathbb{R}$



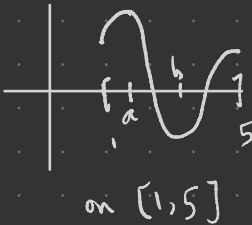
Global maxima at $x = 0, \pm 2\pi, \pm 4\pi, \dots$

Global minima at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \dots$



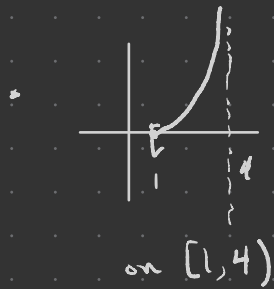
Global max at $x=0$

No global min



Global max at $x=a$

Global min at $x=b$



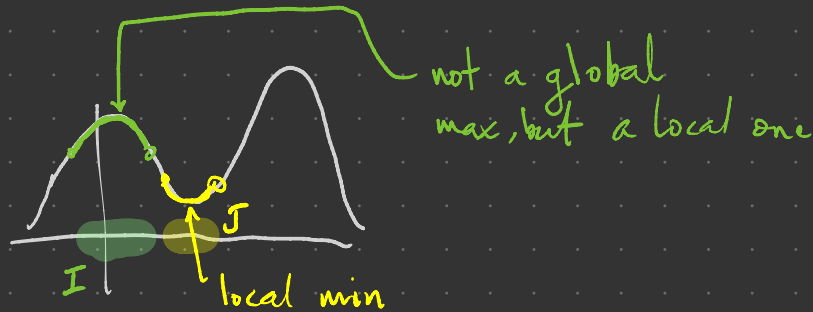
Global min at $x=1$

No global max

on $[1, 4)$

Extreme Value Theorem If f is a continuous function on a closed
bounded interval $[a, b]$, then f has a global max and a global min
on $[a, b]$.

Local extrema



Defn A function f has a local max/min at c if there exists an open interval I containing c on which f has a global max/min at c . These points are called local extrema.

Defn Let c be an interior point in the domain of f . Call c a critical point of f when $f'(c) = 0$ or $f'(c)$ undefined.

horizontal or undefined tangent line at c

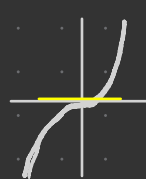
[Fermat]

Thm If f has a local extremum at c , then c is a critical point of f .



Critical point $\nRightarrow$ local extremum.

Consider $f(x) = x^3$ at $x = 0$:


$$f(x) = (x^3)' = 3x^2$$
$$f'(0) = 3 \cdot 0^2 = 0$$



Pf of Thm Suppose f has a local extremum at c . We need to show that f diff'l at $c \Rightarrow f'(c) = 0$.

Assume f has a local max at c (local min case similar).

Take an open interval I containing c on which $f(c) \geq f(x)$ for all $x \in I$.

Since f diff'l at c , $f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ exists.

For $x \leq c$ near c and in I , $f(x) \leq f(c)$ so $f(x) - f(c) \leq 0$

and $x - c < 0$ so $\frac{f(x) - f(c)}{x - c} \geq 0$.

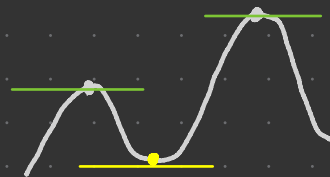
For $x \geq c$ near c and in I , still have $f(x) - f(c) \leq 0$
but $x - c > 0$ so $\frac{f(x) - f(c)}{x - c} \leq 0$ (2)

By (1), $f'(c) = \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \geq 0$.

By (2), $f'(c) = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \leq 0$.

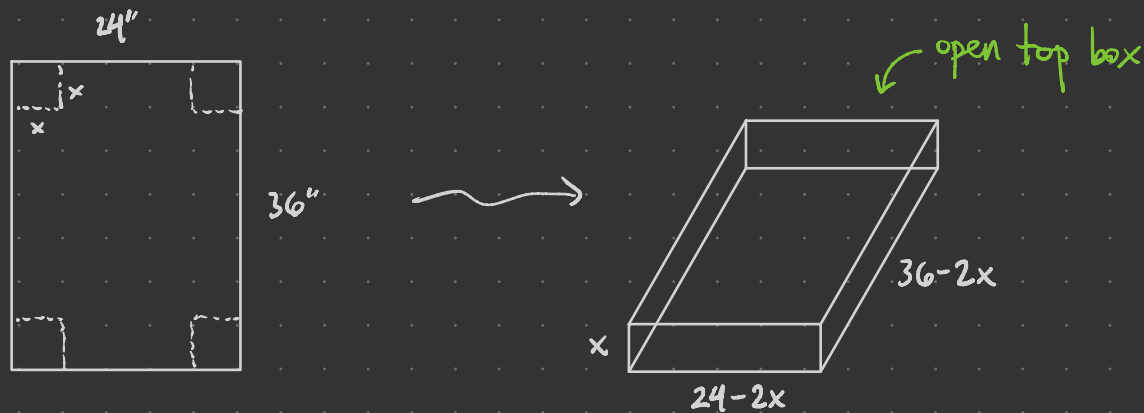
Since $0 \leq f'(c) \leq 0$, $f'(c) = 0$. $\square$

(2) $f'(c) = 0 \not\Rightarrow f$ has a local extremum at c .



To find absolute extrema: compare values at crit pts + endpts.

E.g. Construct a box as follows:



What choice of x maximizes the volume of the box?

$$V = x(24 - 2x)(36 - 2x) = 4x^3 - 120x^2 + 864x \quad \text{in}^3$$

Need to maximize V over $[0, 12]$.

Know max at $x=0$, $x=12$, or a critical pt c where $V'(c)=0$
(or $V'(c)$ undefined).

$$\begin{aligned}\text{Compute } V'(x) &= 12x^2 - 240x + 864 \\ &= 12(x^2 - 20x + 72)\end{aligned}$$

Find c such that $0 = V'(c) = 12(c^2 - 20c + 72)$

Via quadratic formula,

$$\begin{aligned}c &= \frac{20 \pm \sqrt{(-20)^2 - 4 \cdot 1 \cdot 72}}{2 \cdot 1} \\ &= \frac{20 \pm \sqrt{112}}{2} = 10 \pm \frac{1}{2} \sqrt{112}\end{aligned}$$

↑
won't happen
as V is polynomial

$$\begin{array}{r} 72 \\ 12 \overline{) 864} \\ \underline{84} \\ 24 \end{array}$$

$$\begin{array}{r} 72 \\ 4 \\ 2 \overline{) 88} \end{array}$$

$\approx \cancel{15.3}$ or 4.7

not in
 $[0, 12]$

unique crit pt in $[0, 12]$.