

Goals

- Review composition & inverses
- Chain rule: $(f \circ g)'$
- Inverse function theorem: $(f^{-1})'$

Composition

i.e. $(f \circ g)(x) = f(g(x))$ (as long as $g(x)$ is in the domain of f)

For instance, suppose $f(x) = x - 1$ and $g(x) = x^2$.

$$\begin{aligned} \text{Then } (f \circ g)(x) &= f(g(x)) \\ &= f(x^2) \end{aligned}$$



$$= x^2 - 1$$

2 $f \circ g \neq g \circ f$ in general.

With above functions, $(g \circ f)(x) = g(f(x))$

$$= g(x-1)$$

$$= (x-1)^2$$

$$= x^2 - 2x + 1$$

$$x \xrightarrow{z} x^2 + 5 \xrightarrow{w} \frac{1}{x^2 + 5}$$

$$z(x) = x^2 + 5$$

$$w(x) = \frac{1}{x}$$

$$g = w \circ z$$

Problem Express the following functions as compositions:

$$\bullet f(x) = \sin^2 x = (\sin x)^2$$

$$\bullet g(x) = \frac{1}{x^2 + 5}$$

$$x \xrightarrow{k} \sin x \xrightarrow{l} (\sin x)^2$$

$$l(x) = x^2$$

$$k(x) = \sin x$$

$$f = l \circ k$$

$$(\boxed{h} \circ \boxed{g}) \circ \boxed{f} = \boxed{\boxed{h} \circ \boxed{g}} \circ \boxed{f}$$

$$f \circ (g \circ h) = (f \circ g) \circ h$$

$$\boxed{f} \circ \boxed{\boxed{g} \circ \boxed{h}}$$

I.e. function composition is associative

If $f = g \circ h$, then $f(x) = (g \circ h)(x) = g(h(x))$

Inverses The function $\text{id}(x) = x$ is called the identity function.
Its job is to do nothing.

Defn If $f \circ g = \text{id}$ and $g \circ f = \text{id}$, call g the inverse of f
and write $g = f^{-1}$.  $f^{-1} \neq \frac{1}{f}$

Note

- Many functions don't have an inverse.
- Sometimes you need to restrict the domain of a function to specify an inverse.
- The inverse is only defined on the image of f .
- If $f(x) = y$, then $(f^{-1} \circ f)(x) = f^{-1}(y)$
 $\Rightarrow x = f^{-1}(y)$ — inverses solve eq'ns!

E.g. $f(x) = x^2$

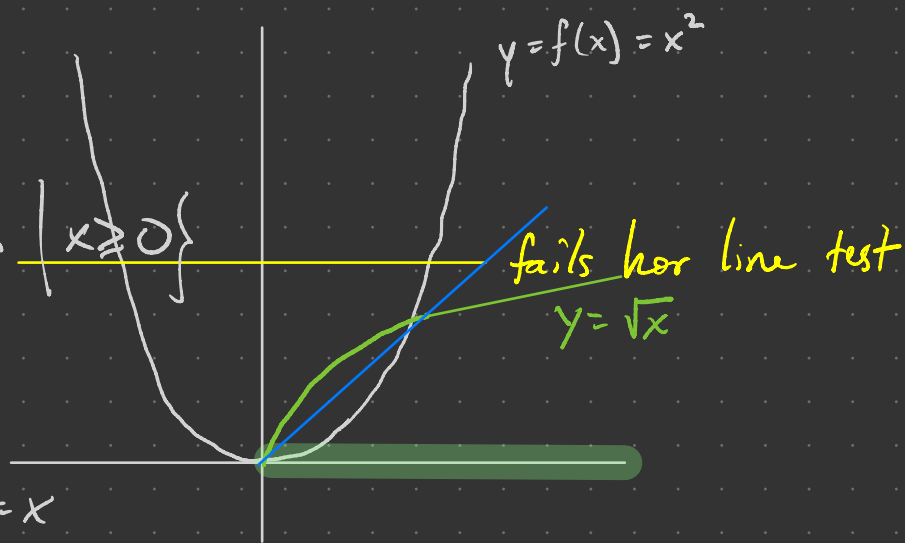
Restrict the domain

of f to $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$

$g(x) = \sqrt{x}$ on $\mathbb{R}_{\geq 0}$.

Then $f(g(x)) = (\sqrt{x})^2 = x$

$g(f(x)) = \sqrt{x^2} = x$



Note Graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ over $y = x$ line.

Chain Rule Let f, g be functions. Then for all x in the domain of g at which (1) g is diff'l, and (2) f is diff'l at $g(x)$,

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) \quad \left| \begin{array}{l} y = g(x) \\ z = f(y) \end{array} \right. \quad \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

E.g. Let's compute the derivative of $h(x) = (2x^3 + 5x^2 - 2x + 3)^4$.

Aha! This $h(x) = f(g(x))$ for $f(x) = x^4$, $g(x) = \underbrace{2x^3 + 5x^2 - 2x + 3}$.

$$\text{Thus } h'(x) = f'(g(x)) \cdot g'(x)$$

$$= 4 \cdot g(x)^3 \cdot (6x^2 + 10x - 2)$$

$$= 4(2x^3 + 5x^2 - 2x + 3)^3 \cdot (6x^2 + 10x - 2)$$

Pf Idea

$$h'(a) = \lim_{x \rightarrow a} \frac{h(x) - h(a)}{x - a}$$

[defn]

$$= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{x - a}$$

[$h = f \circ g$]

$$= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}$$

[mult by
 $1 = \frac{g(x) - g(a)}{g(x) - g(a)}$]

$$= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a}$$

[prod rule
for limits]

$$= \lim_{y \rightarrow g(a)} \frac{f(y) - f(g(a))}{y - g(a)} \cdot g'(a)$$

$g'(a)$

def'n of
deriv

$$= f'(g(a)) \cdot g'(a)$$

□

[defn]

[$y = g(x) +$
as $x \rightarrow a$,
 $g(x) \rightarrow g(a)$]
b/c g cts

Let's use the chain rule to derive the power rule for negative powers from positive powers:

Fix $n > 0$ and set $h(x) = x^{-n} = \frac{1}{x^n} = f(g(x))$ for $g(x) = x^n$, $f(x) = \frac{1}{x}$.

We know $g'(x) = nx^{n-1}$, $f'(x) = \frac{-1}{x^2}$. Thus

$$h'(x) = \frac{-1}{(x^n)^2} \cdot nx^{n-1} \quad [\text{chain rule}]$$

$$= \frac{-n}{x^{n+1}}$$

$$= -n x^{-n-1} \quad \text{— still the power rule!}$$

More generally, $\left(\frac{1}{g(x)}\right)' = \frac{-1}{g(x)^2} \cdot g'(x)$ [chain]

$$= \frac{-g'(x)}{g(x)^2}$$

and $\left(\frac{f(x)}{g(x)}\right)' = \left(f(x) \cdot \frac{1}{g(x)}\right)'$

$$= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \left(\frac{-g'(x)}{g(x)^2}\right)$$
$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \quad \text{--- the quotient rule!}$$