

Goals • Understand the following differentiation rules:

- Constant rule: For c constant,

$$\frac{d}{dx} c = 0$$

- Power rule: $\frac{d}{dx} x^n = n x^{n-1}$

- Linearity: For c a constant, f, g differentiable,

$$\frac{d}{dx} (f + c g) = \frac{df}{dx} + c \frac{dg}{dx}$$

- Product rule: For f, g differentiable,

$$\frac{d}{dx} (f(x) \cdot g(x)) = \frac{df}{dx} \cdot g(x) + f(x) \cdot \frac{dg}{dx}$$

- Quotient rule: For f, g differentiable, whenever $g(x) \neq 0$,

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{df}{dx} - f(x) \cdot \frac{dg}{dx}}{g(x)^2}$$

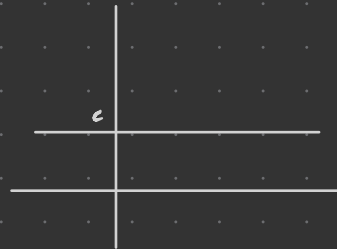
- Constant rule: For c constant, $\frac{d}{dx} c = 0$

Pf For $f(x) = c$,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{c - c}{h}$$

$$= \lim_{h \rightarrow 0} 0 = 0 \quad \square$$



- Power rule : $\frac{d}{dx} x^n = n x^{n-1}$ for any constant n

We will justify the case where n is a positive integer now, n negative integer later, but the general case is true.

Pf for n positive integer By the binomial theorem,

$$(x+h)^n = \underbrace{\binom{n}{0}}_1 h^n + \underbrace{\binom{n}{1}}_n x h^{n-1} + \underbrace{\binom{n}{2}}_{\frac{n(n-1)}{2}} x^2 h^{n-2} + \dots + \underbrace{\binom{n}{n-1}}_n x^{n-1} h + \underbrace{\binom{n}{n}}_1 x^n$$
$$\underbrace{\hspace{15em}}_{h \cdot \left(h^{n-1} + \binom{n}{1} x h^{n-2} + \dots + \binom{n}{n-2} x^{n-2} h + n x^{n-1} \right)}$$

$$\text{so } \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \rightarrow 0} \left(h^{n-1} + \binom{n}{1} x h^{n-2} + \dots + \binom{n}{n-2} x^{n-2} h + n x^{n-1} \right)$$

$$= nx^{n-1} \quad \square$$

E.g.

$f(x)$	$\frac{df}{dx}$
$x = x^1$	$1 \cdot x^0 = 1$
x^2	$2x^1 = 2x$
x^3	$3x^2$
x^4	$4x^3$
x^5	$5x^4$
x^6	$6x^5$
x^7	$7x^6$

But constant fractional
and negative powers work too!

$$\begin{aligned} \frac{d}{dx} \sqrt[3]{x} &= \frac{d}{dx} x^{1/3} \\ &= \frac{1}{3} x^{-2/3} = \frac{1}{3 \sqrt[3]{x^2}} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{x} \right) &= \frac{d}{dx} x^{-1} \\ &= -x^{-2} = -\frac{1}{x^2} \end{aligned}$$

2. Non-constant powers don't work! $\frac{d}{dx}(x^x) \neq x \cdot x^{x-1} = x^x$

$$\frac{d}{dx}(e^x) \neq x e^{x-1}$$

$\equiv e^x$

- linearity: For c a constant, f, g differentiable,

$$\frac{d}{dx}(f + cg) = \frac{df}{dx} + c \frac{dg}{dx}$$

In particular, $(f+g)' = f' + g'$ $c=1$

$$(f-g)' = f' - g' \quad c=-1$$

$$(cf)' = c(f')$$

Pf

$$\begin{aligned}
 \frac{d}{dx}(f + cg) &= \lim_{h \rightarrow 0} \frac{f(x+h) + c \cdot g(x+h) - (f(x) + c \cdot g(x))}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(f(x+h) - f(x)) + c \cdot (g(x+h) - g(x))}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + c \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= \frac{df}{dx} + c \frac{dg}{dx} \quad \square
 \end{aligned}$$

E.g. We can differentiate polynomials!

$$(2x^3 - 5x^2 + x + 3)' = 2(x^3)' - 5(x^2)' + x' + (3)'$$

$$= 2 \cdot 3x^2 - 5 \cdot 2x^1 + 1 \cdot x^0 + 0$$

$$= 6x^2 - 10x + 1$$

Problem Determine $\frac{d}{dx} \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right) = \frac{d}{dx} \left(x^{1/2} - x^{-1/2} \right)$

$$= \frac{d}{dx} \left(x^{1/2} \right) - \frac{d}{dx} \left(x^{-1/2} \right) = \frac{1}{2} x^{\frac{1}{2}-1} - \left(-\frac{1}{2} x^{-\frac{1}{2}-1} \right)$$

$$= \frac{1}{2} x^{-1/2} + \frac{1}{2} x^{-3/2} = \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x^3}}$$

- Product rule: For f, g differentiable,

$$\frac{d}{dx}(f(x) \cdot g(x)) = \frac{df}{dx} \cdot g(x) + f(x) \cdot \frac{dg}{dx}$$

Pf Let $j(x) = f(x)g(x)$. Then

$$j'(x) = \lim_{h \rightarrow 0} \frac{j(x+h) - j(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right)$$

$$\begin{array}{cccc}
 \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} \\
 \text{defn} \downarrow & \downarrow g \text{ cts} & \downarrow \text{lim (corr)} & \downarrow \text{defn} \\
 f'(x) & g(x) & f(x) & g'(x)
 \end{array}$$

$$= f'(x)g(x) + f(x)g'(x), \text{ as desired. } \square$$

E.g. $(x^2)' = (x \cdot x)' = 1 \cdot x + x \cdot 1 = 2x$

$$(x^3)' = (x^2 \cdot x)' = (2x) \cdot x + x^2 \cdot 1 = 3x^2$$

$\vdots$

$$(x^n)' = (x^{n-1} \cdot x)' = (n-1)x^{n-2} \cdot x + x^{n-1} \cdot 1$$

$$= (n-1)x^{n-1} + x^{n-1}$$

$$= nx^{n-1}$$

- Quotient rule: For f, g differentiable, whenever $g(x) \neq 0$,

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{df}{dx} - f(x) \frac{dg}{dx}}{g(x)^2}$$

$$\frac{\text{low} \cdot d(\text{hi}) - \text{hi} \cdot d(\text{low})}{\text{low}^2}$$

We will give a quick proof of this after we see the chain rule!

E.g.

$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{x} \right) &= \frac{x \cdot (1)' - 1(x')}{x^2} \\ &= \frac{-1 \cdot 1}{x^2} \\ &= \frac{-1}{x^2} \end{aligned}$$

E.g. If $h(x) = \frac{3x+1}{4x-3}$, then $h'(x) = \frac{(4x-3) \cdot 3 - (3x+1) \cdot 4}{(4x-3)^2}$

$$= \frac{12x - 9 - (12x + 4)}{(4x-3)^2}$$
$$= \frac{-13}{(4x-3)^2}$$

Problem Where does $y = \frac{x}{x^2 + 1}$ have a horizontal tangent line?