24. IX. 16

Quiz Let x be a real number. Algebraically evaluate

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}.$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{- x^2}}{h}$$ [expand, cancel]

$$= \lim_{h \to 0} \frac{\cancel{h} \cdot (2x + h)}{\cancel{h}}$$ [factor, cancel]

$$= \lim_{h \to 0} (2x + h) = 2x$$ [evaluate at $h=0$]

Goals

- Derivatives as functions
- Sketching derivatives
- Differentiable $\Rightarrow$ continuous

Recall For f a function defined on an open interval containing a,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Defn The derivative function f' is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

with domain those x for which the limit exists.

E.g. From the quiz, if $f(x) = x^2$, then

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= 2x.$$

E.g. If $f(x) = \sqrt{x}$, then

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}.$$

i.e. $(x^{1/2})' = \frac{1}{2} x^{-1/2}$

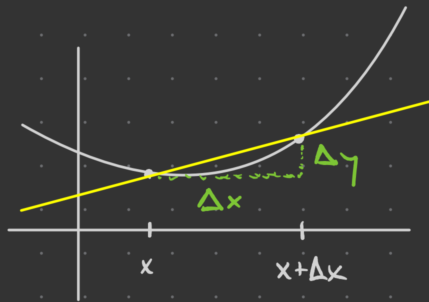
Leibniz notation

Notation $f'(x)$, $\frac{df}{dx}$, $\frac{d}{dx}(f(x))$, and,

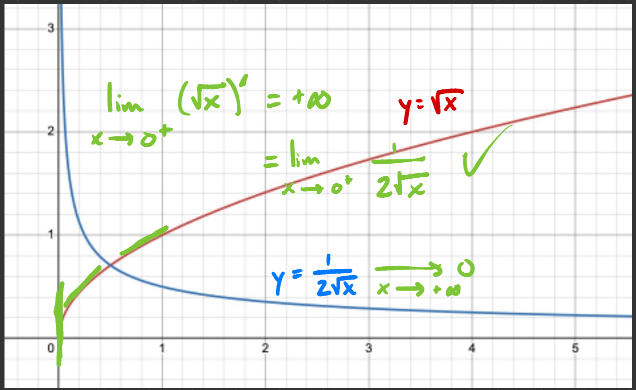
if $y = f(x)$, y' and $\frac{dy}{dx}$ are all notation for the derivative function. To evaluate a deriv in Leibniz notation: $\left.\frac{dy}{dx}\right|_a$

There is also Newton's fluxion/flyspeck notation: $\dot{y}$, $\dot{f}$ etc.

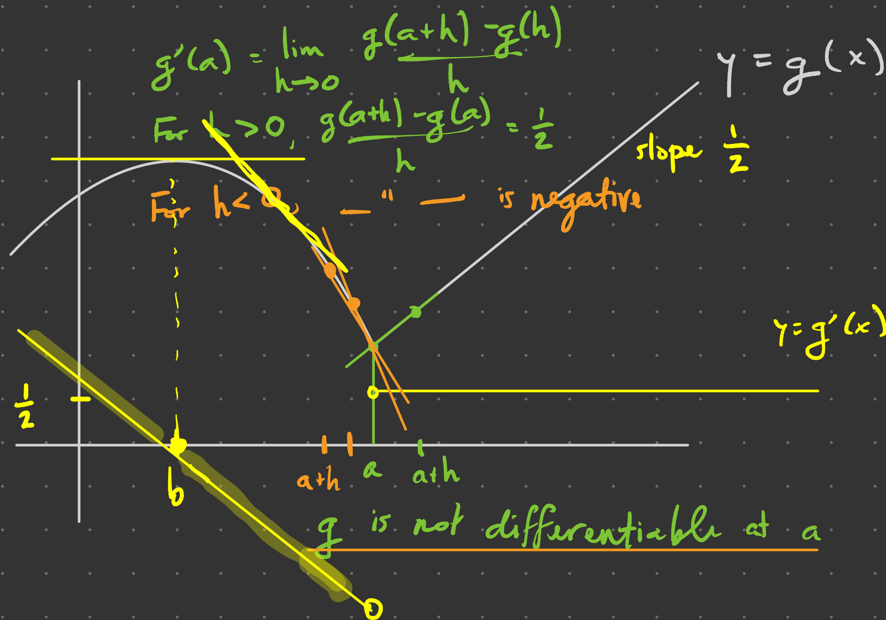
Leibniz notation $\frac{dy}{dx}$ comes from $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$:



Sketching graphs of derivatives



Let's work together to sketch the derivative of this function :



Thm If $f(x)$ is differentiable at $x=a$, then f is continuous at a.

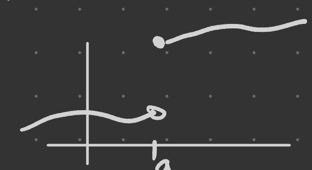
See §3.2 Thm 3.7 for an algebraic proof.

Want to have $\lim_{x\to a} f(x) = f(a)$ given $f'(a)$ exists.

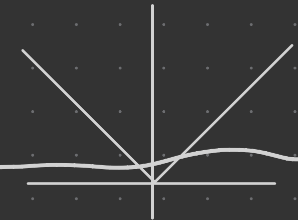
Well, $f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ $\quad x-a \xrightarrow[x\to a]{} 0$ $\quad$ so if $f(x)-f(a) \not\xrightarrow[x\to a]{} 0$

then the limit defining $f'(a)$ does not exist

Thus $f(x)-f(a) \xrightarrow[x\to a]{} 0 \implies f(x) \xrightarrow[x\to a]{} f(a)$.

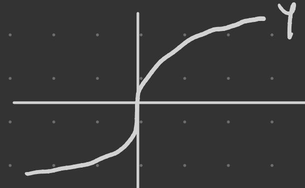


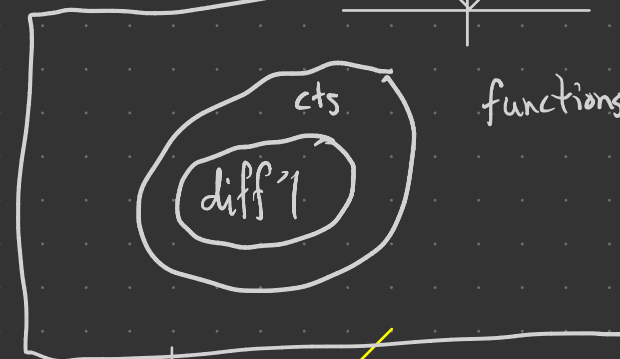
Continuous $\not\Rightarrow$ differentiable



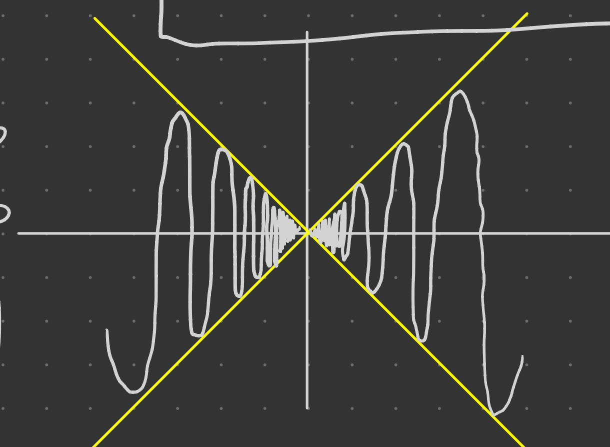
E.g. • $f(x) = |x|$ is cts but not diff'l at $x = 0$

• $f(x) = x^{1/3}$





• $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$



$$f'(0) = \lim_{x \to 0} \frac{x \sin\left(\frac{1}{x}\right) - 0}{x - 0} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

DNE.

Problem Find a, b in $\mathbb{R}$ such that

$$f(x) = \begin{cases} ax+b & \text{if } x<3 \\ x^2 & \text{if } x \geq 3 \end{cases}$$

is both continuous and differentiable at $x=3$

Hint Need $\lim_{x \to 3} ax+b = 3$ and $\left.\frac{dx^2}{dx}\right|_{x=3} = \left.\frac{d(ax+b)}{dx}\right|_{x=3}$

Problem Suppose f, g are differentiable. Explain why $(f+g)' = f' + g'$ and $(af)' = a \cdot (f')$ for any constant a.