

- Goals
- Limit laws
 - Squeeze theorem
 - Continuity

24. IX. 11

Limit laws f, g defined for $x \neq a$ in an open interval containing a , $\lim_{x \rightarrow a} f(x) = L$, $\lim_{x \rightarrow a} g(x) = M$. Take c a constant.

$$\bullet \lim_{x \rightarrow a} (f(x) + g(x)) = L + M$$

limit of a sum = sum of limits

$$\bullet \lim_{x \rightarrow a} c f(x) = c L$$

$$\bullet \lim_{x \rightarrow a} f(x) g(x) = L \cdot M$$

$$\bullet \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ for } M \neq 0$$

$$\bullet \lim_{x \rightarrow a} f(x)^n = L^n \text{ for all positive integers } n$$

Q What about $n < 0$

A $f(x)^{-2} = \frac{1}{f(x)^2}$ if $L \neq 0$, $L^{-2} \text{ DNE}$

• $\lim_{x \rightarrow a} f(x)^{1/n} = L^{1/n}$ for all L if n is odd and for $L \geq 0$

if n is even and $f(x) \geq 0$

E.g. $\lim_{x \rightarrow a} \sqrt{f(x)}$

Q Why the restrictions in the last law? $\left. \begin{array}{l} \text{✓} \end{array} \right\} = \sqrt{\lim_{x \rightarrow a} f(x)}$
for $\nearrow \geq 0$

E.g. $\lim_{x \rightarrow 0} \frac{3x^2 + x - 5}{\sqrt{x+4}}$

Note $x+4 \xrightarrow{x \rightarrow 0} 4$ and $x+4 \geq 0$ near $x=0$, so $\sqrt{x+4} \xrightarrow{x \rightarrow 0} \sqrt{4} = 2$

Also $x^2 \xrightarrow{x \rightarrow 0} 0$, $x \xrightarrow{x \rightarrow 0} 0$, so $3x^2 + x - 5 \xrightarrow{x \rightarrow 0} 3 \cdot 0 + 0 - 5 = -5$.

Thus $\lim_{x \rightarrow 0} \frac{3x^2 + x - 5}{\sqrt{x+4}} = \frac{-5}{2}$

i.e. p is continuous

Thm For p, q polynomials, $\lim_{x \rightarrow a} p(x) = p(a)$

and $\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ for $q(a) \neq 0$.

2 If $p(a) \neq 0$ and $q(a) = 0$, then $\lim_{x \rightarrow a} \frac{p(x)}{q(x)}$ will be infinite or not exist;

if $p(a) = q(a) = 0$, then $\lim_{x \rightarrow a} \frac{p(x)}{q(x)}$ may or may not exist.

E.g. $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2}$

Note that $x^2 + x - 6 = (x - 2)(x + 3)$. Thus

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \rightarrow 2} \frac{\cancel{(x - 2)}(x + 3)}{\cancel{x - 2}}$$

$$= \lim_{x \rightarrow 2} (x + 3)$$

$$= 2 + 3 = 5.$$

If p is a polynomial,
and $p(a) = 0$, then

$$p(x) = (x - a) \cdot \underbrace{q(x)}_{\text{polynomial}}$$

$$x^2 + x - 6 = (x - 2)(x + 3) \quad \checkmark$$

$$\rightarrow ax^2 + bx + c = 0$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

E.g. $\lim_{x \rightarrow -1} \frac{\sqrt{x+2} - 1}{x+1}$ has indeterminate form $\frac{0}{0}$ at $x = -1$.

Trick: Multiply by $\frac{\sqrt{x+2} + 1}{\sqrt{x+2} + 1}$.

$$(y+a)(y-a) = y^2 - a^2$$

Doing so gives

$$\lim_{x \rightarrow -1} \frac{\sqrt{x+2} - 1}{x+1} = \lim_{x \rightarrow -1} \left(\frac{\sqrt{x+2} - 1}{x+1} \cdot \frac{\sqrt{x+2} + 1}{\sqrt{x+2} + 1} \right)$$

$$= \lim_{x \rightarrow -1} \frac{\cancel{x+2} - 1}{(\cancel{x+1})(\sqrt{x+2} + 1)}$$

$$= \lim_{x \rightarrow -1} \frac{1}{\sqrt{x+2} + 1} = \frac{1}{1+1} = \frac{1}{2}$$

Problem Evaluate $\lim_{x \rightarrow 1} \frac{x+2}{(x-1)^2}$

by thinking about

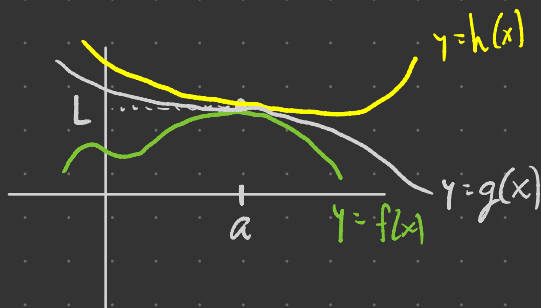
$$\frac{x+2}{(x-1)^2} = (x+2) \cdot \frac{1}{(x-1)^2}$$

$\downarrow$
3 $\downarrow$
 $+\infty$

A

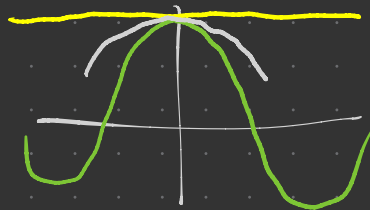
$$\lim_{x \rightarrow 1} \frac{x+2}{(x-1)^2} = +\infty$$

Squeeze Theorem



If $f(x) \leq g(x) \leq h(x)$ near $x=a$ and

$$\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x), \text{ then } \lim_{x \rightarrow a} g(x) = L.$$



$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

By the diagram, for

$$0 \leq x \leq \frac{\pi}{2}$$

$$0 \leq \sin x \leq x \leq \tan x$$

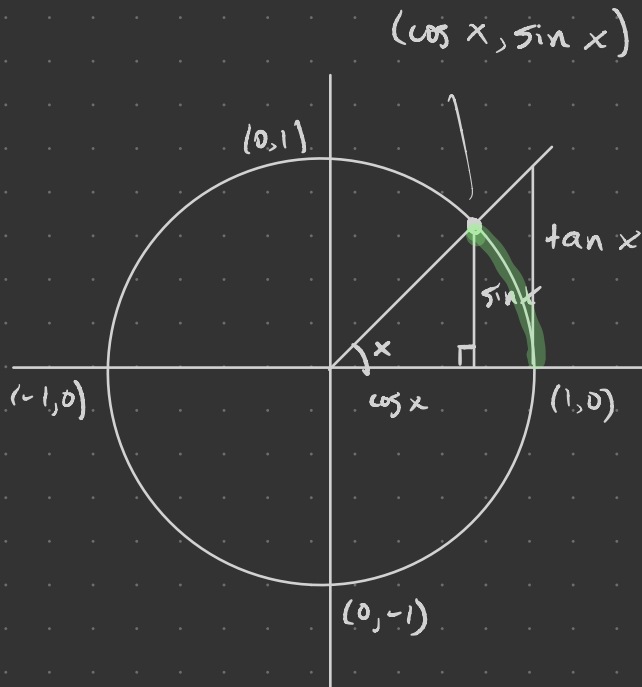
Divide by x :

$$\frac{0}{x} \leq \frac{\sin x}{x} \leq \frac{x}{x}$$

i.e. $0 \leq \frac{\sin x}{x} \leq 1$ Claim $\cos x \leq \frac{\sin x}{x} \leq 1$

$\downarrow x \rightarrow 0$ $\downarrow x \rightarrow 0$
 1 by squeeze!

Got confused here!



x
 arc length
 of sector
 of unit circle
 is x , then
 angle is
 x radians

Fix

As observed, $\sin x \leq x \leq \tan x$.

Dividing by $\sin x$:

$$1 \leq \frac{x}{\sin x} \leq \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x} = \frac{1}{\cos x}$$

Inverting everything (which switches order):

$$\cos x \leq \frac{\sin x}{x} \leq 1.$$

Now squeeze! $\cos x \xrightarrow{x \rightarrow 0^+} 1$ and $1 \xrightarrow{x \rightarrow 0^+} 1$

so $\frac{\sin x}{x} \xrightarrow{x \rightarrow 0^+} 1$. Moral exercise: $x \rightarrow 0^-$ works
as well.